Optimal No-Arbitrage Bounds on S&P500 Index Options and the Volatility Smile *

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This paper shows that the volatility smile is not necessarily inconsistent with the Black-Scholes analysis. Specifically, when transaction costs are present, the absence of arbitrage opportunities does not dictate that there exists a unique price for an option. Rather, there exists a range of prices within which the option’s price may fall and still be consistent with the Black-Scholes arbitrage pricing argument. This paper uses a linear program (LP) cast in a binomial framework to determine the smallest possible range of prices for S&P500 index options which are consistent with no-arbitrage in the presence of transaction costs. The LP method employs dynamic trading in the underlying and risk-free assets as well as fixed positions in other options which trade on the same underlying security. One-way transaction cost levels on the index, inclusive of the bid-ask spread, would have to be below six basis points for deviations from Black-Scholes pricing to present an arbitrage opportunity. Monte Carlo simulations are employed to assess the hedging error induced by using a twelve-period binomial model to approximate a continuous time geometric Brownian motion. Once the risk caused by the hedging error is accounted for, transaction costs have to be well below three basis points for the arbitrage opportunity to be profitable two times out of five. This analysis indicates that market prices that deviate from those given by a constant-volatility option model, such as the Black-Scholes model, can be consistent with the absence of arbitrage in the presence of transaction costs.
1 Introduction

This paper explores deviations from Black-Scholes pricing, commonly known as the ‘volatility smile’ within the context of efficient pricing in the presence of transaction costs. By establishing no-arbitrage bounds using pricing information from related securities and the costs of trading these securities, it is possible to show that the volatility smile may be consistent with constant volatility in that it does not present arbitrage opportunities when transaction costs are present. A Linear Programming (LP) model is used to obtain the tightest possible no-arbitrage pricing bounds for traded options in the presence of transaction costs. The optimization model, originally proposed by Edirisinghe, Naik, and Uppal (1993) and cast in an LP framework by Dennis and Rendleman (1995), incorporates both dynamic trading in the underlying and risk-free asset and fixed positions in other options which trade on the same underlying security. The solution to the LP provides the optimal mix of securities and optimal rebalancing strategy for replicating long and short option positions. I find that the proportional transaction cost on the underlying asset would have to be less than three basis points for the deviations from Black-Scholes pricing to represent a reasonably low-risk arbitrage opportunity to a trader. At transaction costs above this level, no arbitrage opportunities are present.

1.1 The Volatility Smile

Black and Scholes (1973) were the first to develop the strategy of synthetic option replication. Under various assumptions, including perfect markets and no transaction costs, they show that the payoffs to a continuously revised portfolio of stock and risk-free securities can replicate those of a put or call option.

If the underlying asset on which several options are written has constant volatility, and the options are priced according to the Black-Scholes model, then the volatilities which are implied from the market prices of the options using the Black-Scholes model should all be equal to the volatility of the underlying asset. Several studies, including MacBeth and Merville (1979) and Rubinstein (1985) show that not only are the implied volatilities not
equal, but also that the deviations are systematic, and form a “smile” or “sneer” shape. Thus, this variation in implied volatility, known as the volatility smile, must be due to a violation of one or more of the Black-Scholes assumptions.

To explain why the volatility smile exists, several competing theories have been developed, each of which relaxes one or more of the Black-Scholes assumptions. The first set of these theories, such as Rubinstein (1994) and Derman and Kani (1994), relax the constant volatility assumption and allow for a deterministic time and state dependent volatility function. The time and state dependent volatility function allows the returns of the underlying security to have distributions which deviate from the constant volatility normal distribution. In these models, the time and state dependent volatility function is determined by “fitting” the model to the observed market prices of traded options. In this respect these models are similar to the stochastic term structure models of Ho and Lee (1986) and Heath, Jarrow and Morton (1992), in that the model can be adjusted so that the model prices equal the market prices.\footnote{For this to hold in Rubinstein (1994), the time and state dependent volatility function has to be defined with as many free parameters as there are options to price so that there is an exactly identified system of equations.} Dumas, Fleming, and Whaley (1998) provide an empirical test of these time-and-state dependent volatility models and find that there is a large mean square error between the model and market prices. They conclude that the time and state dependent volatility approach does not do a good job of explaining observed option prices.

The second set of these models, such as Hull and White (1987) and Heston (1993), are based on stochastic volatility and can also explain the volatility smile. In contrast with the models of Rubinstein (1994) and Derman and Kani (1994), the volatility of the underlying asset is not deterministic but stochastic. If the volatility has a negative correlation with the underlying security, then when the price of the underlying is low, volatility will be high, hence the probability of large price movements will be high, and the left tail will be fatter than in the case of constant volatility. Conversely, when the price of the underlying is high, volatility will be low, hence the probability of large price movements will be low, and the right tail will be thinner than in the case of constant volatility. Using this type of distribution
to price options will result in relatively high prices for out-of-money puts and relatively low prices for in-the-money puts.

While stochastic volatility models would seem to offer a good explanation for the volatility smile, they do present a problem. In order to be able to price options using no-arbitrage principles, one must either be able to trade a claim on volatility or volatility must be uncorrelated with aggregate consumption.

1.2 Option Replication with Transaction Costs

In addition to assuming that the volatility of an asset is known and is non-stochastic, Black and Scholes also assume that one can trade securities with no transaction costs. Of course, markets are not perfect, and transaction costs do exist. Instead of examining if deviations of option prices from the Black-Scholes model occur due to a non-constant volatility, the approach taken in this paper is to investigate if the deviations are consistent with the absence of arbitrage opportunities in the stock, bond, and options markets, given that transaction costs exist. If one can show that the volatility smile is sandwiched between the long and short no-arbitrage bounds, then the smile may be consistent with the constant volatility Black-Scholes model, once transaction costs have been accounted for.

One of the first studies to examine the impact of proportional transaction costs on the price of an option was Leland (1985). Leland uses a discrete-time stock-bond replication strategy and demonstrates that the payoff of this replication scheme converges almost surely to that of a European call option as the revision interval becomes infinitesimally small. The price of a European call option is determined using the Black-Scholes formula with an inflated variance, $\frac{\sigma^2}{\sqrt{\Delta t}} = \frac{\sigma^2}{\sigma\sqrt{\Delta t}} (1 + \frac{\sqrt{2/\pi k}}{\sigma\sqrt{\Delta t}})$. When the transaction cost, $k$, is large, or the time interval between successive portfolio rebalancings, $\Delta t$, is small, the adjusted volatility is large and the price of the option is relatively high. Though letting large amounts of time lapse between portfolio rebalancings reduces the cost of the option, it also increases the chance that the value of the replicating portfolio will significantly deviate from the desired payoff. While Leland’s approach is limited to derivative securities which have convex payoffs,
Whalley and Wilmont (1993) present an analysis similar to that of Leland (1985) which can be used to replicate concave as well as convex payoffs. Toft (1996) extends Leland’s analysis by examining the mean-variance tradeoff between the variance of the error, which is reduced by rebalancing more frequently, cost, which is increased by rebalancing more frequently.

While Leland (1985) and Whalley and Wilmont (1993) arrive at a closed form solution for the price of a call option, Merton (1990) took a slightly different approach and modeled the stock price as a two-period binomial process. Allowing for proportional transaction costs when trading the stock, Merton then solved for the dynamic trading strategy that exactly replicates the payoff to a call option. Merton’s analysis is extended to binomial trees of arbitrary length in Boyle and Vorst (1992). Using a binomial tree approach is useful for pricing American options, as well as exotics such as Asian options.

Option replication with transaction costs is approached differently in Constantinides (1993). Constantinides models the stock price as a continuous time geometric Brownian motion, uses a proportional transaction cost when purchasing or selling stock, and assumes that the investor has either CRRA or HARA preferences. He then derives the investor’s reservation buy or sell price for a call option on the stock. This analysis is generalized in Constantinides (1996) and Constantinides and Zariphopoulou (1999), where the reservation write or purchase price depends only on an investor’s utility functions being monotonic and concave, not on a specific form of the utility function.

It is important to note that the price of the call option obtained by using the models of Merton (1990), Boyle and Vorst (1992), Leland (1985), Constantinides (1996), Constantinides and Zariphopoulou (1999), and Whalley and Wilmont (1993) is sub-optimal in three respects. First, the stock-bond replicating portfolio has to be rebalanced at each revision date. Due to the frequent rebalancing, the replication cost grows without bound as the revision interval becomes smaller. When transaction costs are present, it is sometimes optimal to not rebalance the replicating portfolio at every single revision date. Second, when transaction costs are present, it is sometimes less expensive to produce a higher payoff than is needed to exactly replicate the call option. In a constrained optimization framework, one can
think of this approach as changing the constraint that the replicating portfolio exactly equals the payoff of the call option to a constraint where the replicating portfolio equals or exceeds the payoff of the call option. Last, these models do not allow for the use of other derivative securities that may be helpful in replicating the call option’s payoff. As will be shown later, using traded puts and calls in the replicating portfolio, in addition to the stock and the bond, helps to lower the cost of replication. The option pricing methodology described in Section 2 overcomes these three problems.

In contrast to these models, Edirisinghe, Naik, and Uppal (1993) and Bensaid, Lesne, Pagès, and Scheinkman (1992) show that the cost of the replicating portfolio may be reduced by following a replicating strategy of dominating an option’s payoff, as opposed to exactly replicating the payoff. Payoff domination is a less expensive replication strategy when transaction costs are high since it may be cheaper to “over-hedge” at certain times, and not to rebalance the stock-bond portfolio at other times.²

While options can be priced using a dynamic stock-bond replicating portfolio and a no-arbitrage argument, it is important to note that stock options can also be priced in a now-and-then economy if markets are complete. Although complete markets are a convenient theoretical construction, they do not exist in practice. In incomplete markets, where the number of states of nature exceeds the number of securities, state prices cannot be implied from observed prices, and stock options cannot be priced by using static replication alone. Nevertheless, Ritchken (1985) uses a linear program to show that the upper and lower no-arbitrage price bounds can be computed for an option using price information from the underlying asset and a risk-free bond, assuming no transaction costs and that the positions in the two securities are maintained without revision until the option’s maturity date.³ In a related work, Cochrane and Saá-Requejo (1996) provide no-arbitrage bounds for option pricing in incomplete markets by choosing a stochastic discount factor to maximize and minimize the price of a call subject to the constraints the stochastic discount factor is positive

²Bensaid, Lesne, Pagès, and Scheinkman (1992) present a simple two-period binomial example which nicely illustrates this point.
³In addition to the price information from the stock and risk-free bond, Ritchken’s method can be easily extended to use the price information contained in other options.
and that its volatility is less than or equal to some pre-specified limit.

Prior empirical work on testing no-arbitrage bounds has focused on put-call parity, which is a special case of static replication. For example, Klemkosky and Resnick (1979) test put-call parity and discover some violations, but they do not account for transaction costs. Nisbit (1992) tests put-call parity and finds that when bid-ask spreads and transaction costs are accounted for, there are no violations. None of these papers, however, has addressed the issue of whether the volatility smile is consistent with the absence of arbitrage opportunities within the context of a static-dynamic replication model. Last, a recent study by Peña, Rubio, and Serna (1999), provides some evidence that the shape of the volatility smile is related to transaction costs.

Given that markets are neither perfect nor complete, bounds for stock option prices can be obtained by using either a dynamic stock-bond replication argument in the presence of transactions costs as in Boyle and Vorst (1992), or an incomplete market static replication argument as in Ritchken (1985), Cochrane and Saá-Requejo (1996), and Garman (1976). Edirisinghe, Naik, and Uppal (1993) and Dennis and Rendleman (1995) combine both these concepts in a linear programming (LP) framework and show that it is possible to obtain even tighter no-arbitrage bounds for the option’s price by simultaneously using both dynamic and static replication. This paper uses the LP methodology to determine upper and lower no-arbitrage prices for the S&P500 index and examines whether the empirically documented volatility smile is consistent with the absence of arbitrage opportunities in the presence of transaction costs.

2 Methodology

2.1 Intuition Behind the Linear Program

This section describes the linear programming (LP) technique used to formulate an optimal option replication strategy. The objective of the LP is to determine the lowest cost portfolio consisting of stock, a risk-free bond, and traded options that replicates the payoffs to a
non-traded option. The LP combines both dynamic and static replication techniques. An example is provided here to convey the intuition. A detailed description of the model is provided in Appendix 1, as well as in Edirisinghe, Naik and Uppal (1993) and Dennis and Rendleman (1995).

Suppose that there exists a risky asset which has a current price of $100 and a volatility of 20% per annum, and our goal is to replicate a one-year call with a strike price of $100. The evolution of the stock price is modeled as a two-period binomial tree with one-half year per period and is shown in Figure 1.

In addition to the stock, there also exists a risk-free asset, which has a current value of $1 and which earns a continuous return of 10% per annum. Its value at the end of the first period is $1.0513 and its value at the end of the second period is $1.1052. Finally, there is an one-year call option available with a striking price of $80, which currently costs $28. A 1% transaction cost is incurred whenever the stock, the risk-free asset or the call are traded.

Our objective is to minimize the up-front portfolio cost, which, using equation (16), and accounting for the 1% transaction cost, can be expressed as:

$$101\Delta_{0,0}^{LA} - 99\Delta_{0,0}^{SA} + 1.01\Delta_{0,0}^{LB} - 0.99\Delta_{0,0}^{SB} + 28.28\Delta_{0,0}^{LO} - 27.72\Delta_{0,0}^{SO}$$

(1)

where $\Delta_{i,j}^{x}$ represents the change in the quantity of asset $x$ held long (L) or short (S) after the stock has moved up $i$ times and down $j$ times.

The first set of constraints that must be specified are the linking constraints, which link the total position of the stock and risk-free asset in each state to the changes in the positions of these assets between adjacent states. For this problem these constraints are:

$$\Phi_{0,0}^{LA} - \Phi_{0,0}^{SA} = \Delta_{0,0}^{LA} - \Delta_{0,0}^{SA}$$

(2)

$$\Phi_{0,0}^{LB} - \Phi_{0,0}^{SB} = \Delta_{0,0}^{LB} - \Delta_{0,0}^{SB}$$

(3)

$$\Phi_{0,0}^{LO} - \Phi_{0,0}^{SO} = \Delta_{0,0}^{LO} - \Delta_{0,0}^{SO}$$

(4)

$$\Phi_{1,0}^{LA} - \Phi_{1,0}^{SA} = \Phi_{0,0}^{LA} - \Phi_{0,0}^{SA} + \Delta_{1,0}^{LA} - \Delta_{1,0}^{SA}$$

(5)

$$\Phi_{1,0}^{LB} - \Phi_{1,0}^{SB} = \Phi_{0,0}^{LB} - \Phi_{0,0}^{SB} + \Delta_{1,0}^{LB} - \Delta_{1,0}^{SB}$$

(6)
\[ \Phi_{0,1}^{LA} - \Phi_{0,1}^{SA} = \Phi_{0,0}^{LA} - \Phi_{0,0}^{SA} + \Delta_{0,1}^{LA} - \Delta_{0,1}^{SA} \]  
\[ \Phi_{0,1}^{LB} - \Phi_{0,1}^{SB} = \Phi_{0,0}^{LB} - \Phi_{0,0}^{SB} + \Delta_{0,1}^{LB} - \Delta_{0,1}^{SB} \]

Constraint (2) sets the total initial position of the stock, \( \Phi_{0,0}^{LA} - \Phi_{0,0}^{SA} \), equal to the amount of stock initially purchased, \( \Delta_{0,0}^{LA} - \Delta_{0,0}^{SA} \). These two quantities must be the same, since there was no stock already owned at the beginning of the replication program. Constraints (3) and (4) apply to the risk-free asset and traded option, respectively, and parallel constraint (2). Constraint (5) sets the total amount of stock owned in state \( \{1, 0\} \), \( \Phi_{0,0}^{LA} - \Phi_{0,0}^{SA} \), equal to the amount of stock owned in state \( \{0, 0\} \), \( \Phi_{0,0}^{LA} - \Phi_{0,0}^{SA} \), plus the amount purchased in state \( \{1, 0\} \), \( \Delta_{1,0}^{LA} - \Delta_{1,0}^{SA} \). Constraint (6) is identical to constraint (5), except that it applies to the risk-free asset. Constraints (7) and (8) for state \( \{0, 1\} \) parallel constraints (5) and (6) for state \( \{1, 0\} \).

In addition to the linking constraints, the portfolio must be self-financing in state \( \{1, 0\} \) and state \( \{0, 1\} \). Before we can define the self-financing constraints, we must remember to account for transaction costs. For example, the cost of purchasing the stock in state \( \{1, 0\} \) is \((115.19)(1.01) = 116.34\), and the cash inflow from shorting the stock is \((115.19)(0.99) = 114.04\). The following two constraints ensure that the replicating portfolio is self-financing in both the up and down states at time 1:

\[ 116:34\Delta_{1,0}^{LA} - 114:04\Delta_{1,0}^{SA} + 1:0618\Delta_{1,0}^{LB} - 1:0408\Delta_{1,0}^{SB} + S_{1,0}^+ = 0 \]  
\[ 87:68\Delta_{0,1}^{LA} - 85:94\Delta_{0,1}^{SA} + 1:0618\Delta_{0,1}^{LB} - 1:0408\Delta_{0,1}^{SB} + S_{0,1}^- = 0 \]

Equation (9) constrains the replicating portfolio to be self-financing in state \( \{1, 0\} \). The constraint ensures that the cost of purchasing stock at 116.34, less the cash inflow generated from shorting stock at 114.04, plus the cost of purchasing the risk-free asset at 1.0618, less the cash inflow of shorting the asset at 1.0408 is less than or equal to zero. Constraint (10), which applies to state \( \{0, 1\} \), parallels constraint (9), which applies to state \( \{1, 0\} \). The variables \( S_{1,0}^+ \) and \( S_{0,1}^- \) are non-negative slack variables which transform constraints (9) and (10) from inequalities into equalities.

To complete the LP, we must specify the constraints which ensure that, at maturity, the
payoffs of the replicating portfolio meet or exceed those of the target. The following four constraints accomplish this:

\[ 131.36 \Phi^{LA}_{1,0} - 134.02 \Phi^{SA}_{1,0} + 1.0941 \Phi^{LB}_{1,0} - 1.1162 \Phi^{SB}_{1,0} + 52.16 \Phi^{LO}_1 - 53.22 \Phi^{SO}_1 \]
\[-S^+_{2,0} = 32.69 \quad (11)\]

\[ 99.00 \Phi^{LA}_{1,0} - 101.00 \Phi^{SA}_{1,0} + 1.0941 \Phi^{LB}_{1,0} - 1.1162 \Phi^{SB}_{1,0} + 19.80 \Phi^{LO}_1 - 20.20 \Phi^{SO}_1 \]
\[-S^-_{1,1} = 0 \quad (12)\]

\[ 99.00 \Phi^{LA}_{0,1} - 101.00 \Phi^{SA}_{0,1} + 1.0941 \Phi^{LB}_{0,1} - 1.1162 \Phi^{SB}_{0,1} + 19.80 \Phi^{LO}_{0,1} - 20.20 \Phi^{SO}_{0,1} \]
\[-S^+_{1,1} = 0 \quad (13)\]

\[ 74.61 \Phi^{LA}_{0,1} - 76.12 \Phi^{SA}_{0,1} + 1.0941 \Phi^{LB}_{0,1} - 1.1162 \Phi^{SB}_{0,1} + 0 \Phi^{LO}_{0,1} - 0 \Phi^{SO}_{0,1} \]
\[-S^-_{0,2} = 0 \quad (14)\]

Constraint (11) applies to state \( \{2, 0\} \). The positive terms in (11) represent the after-transaction-cost cash inflow which is generated at time 2 from liquidating the long position in the stock at $131.36, the risk-free asset at $1.0941, and the call option at $52.16. The negative terms in (11) represent the after transaction cost cash outflows from covering the short position in the stock at $134.02, the risk-free asset at $1.1162, and the call option at $53.52. Since \( S^+_{2,0} \) is a non-negative slack variable, the total cash inflows must exceed the total cash outflows by at least $32.69, which is the target replicating portfolio value in state \( \{2, 0\} \). While constraint (11) ensures that the target value will be met or exceeded when the stock increases from state \( \{1,0\} \) to state \( \{2,0\} \), constraints (12), (13), and (14) ensure that the target value will be met or exceeded when the stock price moves from state \( \{1,0\} \) to \( \{1,1\} \), state \( \{0,1\} \) to \( \{1,1\} \), and state \( \{0,1\} \) to \( \{0,2\} \), respectively.

The objective function (1), along with the constraints (2) to (14), constitute a linear program whose solution represents the optimal cost and optimal trading strategy for replicating a call option. In this particular case, the objective function is $14.16 at the optimum.
3  Data

The primary data set used is a subset of the Berkeley Options Data Base which contains both quotes and trades recorded on the floor of the CBOE from January 4, 1993 to April 30, 1993. Each record is time stamped to the nearest second and contains the bid-ask quote, ticker symbol, date, time, type (put or call), strike price, and the latest value of the underlying security. The options used are S&P500 index options, which are cash settled and European in nature. The term structure at each point in time is constructed using bid and ask T-bill rates collected from The Wall Street Journal.

There are two concerns that must be addressed when constructing the subset of options to be used for the static component of the replication - the depth of the market and the synchronicity of the observed option prices. First, it is important to have enough market depth so that trading in the options does not move the price significantly. Since most of the trading in index options occurs in the near-the-money options, only the three closest-to-the-money calls and puts are used in the replicating portfolio, for a maximum of six options. For ease of exposition, let Z represent the set of six options that meets the liquidity criteria described above.

In addition to the liquidity criteria, it is also important that the option prices are observed at the same point in time. While it would be ideal to observe all option prices at the same point in time, $t^*$, in practice the prices are usually quoted at different points in time, and this non-synchronicity can introduce error. To minimize this error, only those options in set $Z$ that are closest to time $t^*$, without being more than five minutes away, are used in the static component of the replication.

Once the set of options to be used for the static component of the replication has been selected, the next step is to construct the binomial tree which will be used to determine the dynamic part of the replication. When constructing the binomial tree, there are three issues which must be addressed - the treatment of dividends, the determination of the initial index level, and the determination of the volatility of the index.

Dividends on the S&P500 stock index must be treated with care. Harvey and Whaley
(1992) show that there exist strong seasonal patterns in the dividend payouts for the S&P100 index, with February, May, August, and November having the highest dividend payouts. They also show that there is a strong day-of-the-week effect, with Monday having the highest dividend payout, and Wednesday having the lowest. Due to these effects, dividends should not be treated as continuous but rather as discrete. Daily cash dividends for the S&P 500 index were obtained from the S&P500 Information Bulletin. Though the dividends are discrete, the binomial tree was adjusted for discrete dividends in such a way that it re-combines.

To model the evolution of the index value using a binomial tree, both the initial level of the index and the volatility of the index must be known. While each record in the Berkeley Options Database contains the current index level, it may not represent the true level of the underlying index. This discrepancy arises because the index is a composite of prices of 500 stocks, some of which do not trade that frequently. Due to this “stale quote” problem, the quoted index level may not reflect the true market consensus of what the index value is. To mitigate this stale index problem, both the volatility and the level of the index are jointly implied from those options that meet both the liquidity and synchronicity criteria described above by minimizing the sum-of-squared deviations between the market prices of the options and their model prices.

4 Results
4.1 The Volatility Smile

While full empirical tests are discussed in Section 4.2, this section gives an example of how the volatility smile can be consistent with the absence of arbitrage opportunities in the presence of transactions costs. As long as the volatility for the traded option falls between that of the synthetic long and short positions, there will be no arbitrage opportunities involving the

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4 Robert E. Whaley graciously provided these data.
6 The model prices are computed using the dividend-adjusted binomial tree and the term structure implied by Treasury Bill prices from in The Wall Street Journal.
target security and any combination of securities in the replicating portfolio. If the volatility smile is sandwiched between the implied volatility bounds formed by the long and short synthetic positions, we can conclude that the volatility smile is consistent with the absence of arbitrage opportunities, once we have accounted for transaction costs.

Table 1 shows the market prices on January 1, 1993 at 2 p.m.. A proportional transaction cost of 0.5% is used for the underlying index, and no transaction cost, other than the bid-ask spread, is used for the index options and the risk-free asset. The transaction cost of 0.5% was chosen simply to illustrate this example. In the empirical tests in the next section, the transaction cost is continuously lowered until an arbitrage is possible. This level of transaction cost can then be compared to actual costs that traders incur to see if arbitrage is feasible at realistic transaction cost levels.

Table 2 shows the no-arbitrage volatility bounds computed for February calls with striking prices ranging from 420 to 450. The prices for the synthetic calls in the table are constructed using the data for the stock, bond, and options 1 through 5 from Table 1. The upper panel and lower panel show replicating costs and implied volatilities for long and short positions, respectively. In each panel, the first column lists strike prices from 420 to 450. The second column contains the synthetic price of a long call using the stock, bond, and options 1 through 5 from Table 1. The third column contains the bid price for the option which was quoted closest to 2:00 P.M.. The fourth and fifth columns contain the implied volatility for the prices in the second and third columns.

There are several observations that can be made about the upper panel of Table 2. First, the volatility smile is evident in the observed bid prices. The implied volatility is 11.25% at a strike of 420, and declines to a value of 10.33% at a strike of 450. Second, at most strike prices, the cost to synthetically create a long position is greater than the cash inflow that would be realized by shorting a traded option at the bid price. For example, it would cost $18.222 (implied volatility of 12.70%) to synthesize a long call struck at 420, but only $17.63 (implied volatility of 11.25%) could be generated by shorting the option in the market. Hence, there is no arbitrage opportunity. The one exception to this is the case where the
strike price is 445. Here the synthetic cost of a long call is $2.870$, but the call has a bid price of $2\frac{7}{8}$ (2.875). This implies that even after a transaction cost of 0.5%, an arbitrage profit is possible. This may not be a true arbitrage opportunity, however, for two reasons. First, the option is so far out of the money that one may not be able to short the option at $2\frac{7}{8}$ due to a thin market. Second, even if one can short the option at $2\frac{7}{8}$, there will be error introduced into the hedge, since the binomial model is only an approximation to the actual stochastic process followed by the stock. As long as the stock’s price evolves according to the binomial model, the LP replication will be exact. If, however, the stock’s price path deviates from the binomial lattice at some future date, hedging error will be introduced. The impact of this hedging error on the transaction cost necessary for arbitrage opportunities to exist is discussed in Section 4.3.

The lower panel of Table 2 is similar to the upper panel, except that it contains the costs of synthetic short positions and the ask prices of the traded options. As in the upper panel, the volatility smile is evident in the observed ask prices. Also, at each strike price the cost to purchase a long option in the market is greater than the cash inflow that would be realized by synthetically creating a short position. Hence, the volatility smile is bounded below by the implied volatilities of the synthetic short positions. Since the volatility smile is bounded above by the implied volatilities of the synthetic long positions (except for the case where the strike is 445), and is bounded below by the implied volatilities of the synthetic short positions, the volatility smile is consistent with the absence of arbitrage opportunities, once transaction costs have been accounted for.

### 4.2 Transaction Costs Necessary to Permit Arbitrage

This section discusses the minimum level of transaction costs such that the deviations from Black-Scholes model are still consistent with the constant volatility model and the absence of arbitrage. To determine these costs, the analysis outlined in Section 4.1 could be repeated at successively lower transaction costs until an arbitrage opportunity is detected. This would be extremely time consuming, as the LP would have to be re-run several times at each
level of transaction costs to re-create the bounds on the smile. An alternative approach to
detecting arbitrage opportunities is to re-formulate the LP to minimize the up-front cost of
the replicating portfolio subject to the constraint that a non-negative payoff be generated at
each state at the maturity of the replicating portfolio. If a portfolio can be found that satisfies
the non-negativity constraint at the replicating portfolio’s maturity, and has a negative price
today (i.e., a cash inflow), then an arbitrage opportunity has been detected. At high levels
of transaction costs no arbitrage opportunities exist. As transaction costs fall, a point will
be reached where an arbitrage opportunity is possible. This method is more efficient, since
the LP only has to be re-run once at each transaction cost level.

Table 3 shows the results from 20 replicating portfolios that were constructed at 1 P.M.
on each trading day during January, 1993, and the transaction cost levels at which arbitrage
opportunities exist. The options in the replicating portfolio were selected according to the
liquidity and synchronicity criteria outlined in Section 3. There is no transaction cost placed
on the risk-free asset, other than the bid-ask spread obtained from The Wall Street Journal,
and there is no transaction cost placed on the traded options, other than the bid-ask spread
obtained from the Berkeley Options Database. The level of transaction cost at which arbi-
trage opportunities exist is determined by setting the initial transaction cost on the index to
a very high level and lowering it until an arbitrage opportunity is detected. This transaction
cost will then be compared to the bid-ask spread and commission costs for trading the index.

The first three columns in Table 3 list the date that the replicating portfolio was con-
structed, the number of days to maturity for the replicating portfolio, and the break-even
transaction cost. The break-even transaction cost represents the largest cost at which arbi-
trage is feasible within the context of the binomial model. For example, if, on January
4, 1993, a trader incurred a one-way transaction cost of 3.78 basis points or lower, then
there would be an arbitrage opportunity. The average transaction cost at which arbitrage
is feasible is roughly 6 basis points. It is useful to compare these costs to costs that would
actually be incurred by traders. The typical bid-ask spread was around two to three ticks

7Varying the transaction cost on all three types of securities complicates matters, since there would be
three choice variables instead of one.
(0.10 to 0.15) on S&P 500 index futures in 1993 and the commission per contract was roughly $30. This gives a one-way transaction cost of around $\frac{(500)(0.05)(2.5/2)+30}{(430)(500)}$ or roughly three basis points per trade. To see if the results are robust through time, the arbitrage analysis presented above was repeated for observations during January 1995. The results are presented in Table 4. The results for January 1995 indicate that the average transaction cost at which arbitrage is feasible is roughly five basis points, which is on the same order of magnitude as the transaction cost for January 1993.

While it appears from this data that arbitrage seems to be economically feasible, a trader may not be able to take advantage of these opportunities. Since the model is a binomial approximation to the continuous time process being used, these arbitrage positions may not be entirely risk-free. As will be discussed in Section 4.3, the transaction costs have to be much lower than three basis points to have a reasonably low-risk arbitrage opportunity.

### 4.3 Hedging Effectiveness

In the previous section, it was shown that arbitrage opportunities could be detected by constraining the replicating portfolio to produce a non-negative payoff in each state, and then gradually lowering the transaction cost levels to discover at what point the up-front replicating portfolio has a negative price. One of the problems with this is that the LP model is necessarily based on the assumption that the price of the underlying stock follows a binomial process, when, in fact, the binomial model is being used as an approximation to a continuous lognormal distribution. The arbitrage trading is guaranteed to work perfectly if the actual stock price outcomes fall exactly on the nodes of the binomial tree. If, however, the stock price outcomes do not fall exactly on the binomial tree, as will be the case if the true stock price is log-normally distributed, any arbitrage opportunities that are detected may not be risk-free. In this section, Monte Carlo simulations are used to determine the magnitude of the risk introduced by the binomial approximation.

It is important to recognize that the LP solution based on the binomial model not only

---

8 This figure was obtained from the CME.
determines the initial set of portfolio holdings for the replicating portfolio, but also the optimal holdings in each subsequent binomial state. In the simulations it is assumed that the investor expects stock returns to follow binomial outcomes and also expects to re-solve the LP at the end of each binomial period based on future anticipated binomial outcomes. In fact, this procedure is implicit in the optimal binomial based LP solution. Actually, however, stock return outcomes are generated from a normal distribution. At the end of the first binomial period, the actual stock price is likely to be different from either of the two binomial outcomes that had been originally anticipated and reflected in the initial LP solution. As such, the LP, based on \( T - 1 \) remaining periods, must be re-solved using the simulated stock price realized at the end of the first binomial period, and using the security positions carried forward from time zero. Since the binomial model is being used as an approximation of a continuous time stochastic process, most of the time the random return will not fall exactly on the “up” or the “down” nodes of the binomial tree. Hence, the replicating portfolio will most likely not be self-financing, and cash will have to be added to, or withdrawn from, the replicating portfolio at the end of the first binomial period.

Having solved for the optimal replicating portfolio at the end of the first period, a random return for the stock is then realized at the end of the second binomial period, a \( T - 2 \) period binomial tree is constructed, and the process is repeated. This algorithm is continued until the maturity of the option being replicated is reached. The sum of the future value of the cash infusions or withdrawals at each time represents a hedging error for some price path realization from the continuous time stochastic process. By drawing \( M \) random sample paths from the true stochastic process and computing the hedging error for each sample path, the hedging precision of the linear programming model, defined as the standard deviation of the future value of the \( M \) hedging errors, can be determined. To generate a series of normally distributed random returns for the stock, uniform random deviates are generated using the RAN2 routine from Press, Teukolsky, Vetterling, and Flannery (1992). This random number generation routine provides random numbers which have a periodicity of about \( 2.3 \times 10^{18} \). The uniform random deviates are transformed into normal random numbers via the Box-
Muller transformation by the GASDEV routine from Press (1992).

The initial replicating portfolio, which may consist of stock, bonds, and options, is constrained to produce a zero payoff in each of the thirteen possible states at the end of the binomial lattice. It is important to note that if the LP is used as it is formulated in Section 2.1, an arbitrage opportunity will result in an unbounded objective function.\(^9\) If the purpose of the LP is only to detect the presence of an arbitrage opportunity, as in the prior section, then the fact that the solution is unbounded does not matter. If one is interested in the composition of the optimal arbitrage portfolio, however, then the solution to LP cannot be unbounded. This is prevented by adding a constraint to the LP to ensure that the value of the objective function cannot be lower than -1.00, thereby ensuring that the solution to the LP will be bounded. This constraint also standardizes the initial arbitrage profit to be $1.00 for varying levels of transaction costs, since, when an arbitrage opportunity is present, the constraint will be binding.

Log-normally distributed stock price outcomes are generated by starting with the initial value of the asset and generating subsequent values for the asset using:

\[
S_{t+1} = S_t e^{(r_f \tau + \sigma \tilde{z} \sqrt{\tau})}
\]

where \(\tilde{z} \sim N (0; 1)\), and \(r_f\) is the risk-free rate, determined from the midpoint of the T-bill bid-ask spread, observed at time \(t\). The volatility used for the asset’s returns is the implied volatility described in Section 3. The transaction cost on the underlying asset is taken from column 3 in Table 3, which is the largest cost where arbitrage is still possible.

The fourth and fifth columns of Table 3 contain the mean and standard deviation of the hedging error after 100 simulated price paths. The magnitude of the standard deviation of the hedging error is quite large. To understand this, one must consider that to realize $1 of profit at the largest transaction cost level where arbitrage is still possible, very large positions must be taken in the underlying securities. This is because transaction costs consume most of the profit. Typical initial positions are shown in Table 5. The first column shows the positions that must be taken when the transaction costs are just below the break-even point. As can

\(^9\)To see this, say that there exists some portfolio which satisfies the constraints, and has a value of \(-\pi\) today. Doubling the portfolio positions will result in a value today of \(-2\pi\), yet still satisfy all the constraints. Since the objective is to minimize the up-front portfolio cost, the value of the objective function at the optimum will be \(-\infty\).
be seen from the first column, the magnitudes of the positions are quite large. Considering that a volume of 10,000 for an S&P500 index option is pretty heavy, there is not even enough liquidity to make some of the positions feasible at the observed prices.

The last column of Table 3 contains the ratio of the mean error to the standard deviation. Though the replicating strategy produces a positive mean hedging error, the magnitude of the mean hedging error is small when compared to the magnitude of the standard deviation, indicating that there is a good chance of large losses. The first row of Table 6 summarizes the data in Table 3 and contains two additional statistics. The first is the percentage of time that the hedging error is negative. In the case where the transaction cost is just below the break-even point, the arbitrage strategy loses money roughly half of the time. Furthermore, conditional on the hedging error being negative, the average dollar loss is $1,629. This is a large risk to make $1 in arbitrage profit.

It is natural to ask if the situation can be improved if a trader incurs a lower transaction cost. Two observations can be made. First, the size of the positions required for arbitrage is lower at the reduced transaction cost. Table 5 shows the positions required for arbitrage on a typical day when transaction costs are 10% and 50% below the break-even level. While relatively smaller positions are required at these lower transaction cost levels, they are still large compared to the typical daily volume in the S&P500 index options market. Second, while the arbitrage transactions are relatively less risky at these lower transaction cost levels, they are still quite risky on an absolute basis. The second row of Table 6 shows the summary statistics for 100 price path simulations on each day of January 1993. The results were obtained in the same manner as those in the first row of the table, except that the transaction cost used for each of the simulations is 10% less than the break-even transaction cost for that particular day. While the standard deviation of the error is reduced dramatically, it is still much larger than the mean. The percent of time that a trader loses money does not change and is still around 50 percent. Furthermore, the dollar loss conditional on the hedging error being negative is $867, which is still quite large compared to the $1 of arbitrage profit. The last row contains the results when the transaction cost is lowered by 50% from the break-even
value to three basis points. Even at this level of transaction costs, there is still a large risk, with a loss occurring 43% of the time and averaging $257.

This analysis suggests that while arbitrage may at first appear feasible, it is by no means risk-free. Though an arbitrage opportunities were detected at a transaction cost level averaging six basis points, the positions in the replicating portfolio have to be so large to realize $1 in profit that a small amount of model mis-specification can induce very large errors. Even as the transaction costs are lowered to 50% less than the break-even value, any attempted arbitrage is still very risky.

5 Conclusion

This paper shows that the deviations of option prices from those given by a constant-volatility option model, such as the Black-Scholes model, can be consistent with the absence of arbitrage in the presence of transaction costs. Specifically, a linear program is used to determine the optimal dynamic and static replication method to synthesize long and short positions for an S&P 500 index option. As long as the bid price is above the synthetic short price, and the ask price is below the synthetic long price, there is no way to make an arbitrage profit. If, however, either of these conditions is violated it is possible to make an arbitrage profit, though the arbitrage opportunity may not be entirely risk-free. The use of both dynamic and static replication allows for stronger tests of the existence of arbitrage opportunities than previously possible.

Using a twelve-period binomial model to capture the price dynamics of a continuous time process, one-way transaction costs, inclusive of the bid-ask spread, would have to be lower than six basis points for the volatility smile to be inconsistent with the absence of arbitrage. While an arbitrage opportunity may be detected at this transaction cost level, it may not be risk-free due to model mis-specification. Monte Carlo simulations are used to assess the risk induced by using a binomial tree to approximate a continuous time process. Though an arbitrage opportunity may be present at a transaction cost level of six basis points, such large positions have to be taken to realize a $1 profit that the arbitrage is very risky. Even at
transaction cost levels lower than 3 basis points, which is roughly what traders would incur, there are no reasonably risk-free arbitrage opportunities.
References


6 Appendix: LP Details

This appendix describes the linear programming technique used to formulate an optimal option replication strategy. The objective of the LP is to determine the lowest cost portfolio consisting of stock, a risk-free bond, and traded options that replicates the payoffs to a non-traded option. The LP combines both dynamic and static replication techniques.¹⁰

The uncertainty in the future stock price is represented using a binomial model as in Cox, Ross, and Rubinstein (1979) and Rendleman and Bartter (1979). Figure 2 shows a three-period binomial price process. Let \(A_{0,0}\) denote the current price of the underlying stock and \(A_{i,j}\) denote the price of the underlying stock after it has increased in value \(i\) times and decreased in value \(j\) times. Let \(\theta\) represent the length of one binomial period in years, \(T\) represent the total number of binomial periods, and let \(\frac{\sigma}{\sqrt{T}}\) represent the annual volatility of the stock’s returns. The future price of the stock after \(i\) ups and \(j\) downs is given by \(A_{i,j} = A_{0,0}e^{(i-j)\frac{\sigma}{\sqrt{T}}}.\) Assume that a risk-free asset exists that has an annual continuously compounded return of \(r\) per year. Let its value at time zero be denoted as \(B_{0,0}\) and its value in state \(i,j\) be denoted as \(B_{i,j}.\) The future value of the price of the risk-free asset is given by \(B_{i,j} = B_{0,0}e^{r(i+j)\theta}.\)

Though the LP methodology can accommodate both fixed and proportional transaction costs, the model described here will focus on proportional costs only. Define \(\frac{\gamma}{A}\) as the proportional cost of trading the underlying stock, \(\frac{\gamma}{B}\) as the proportional cost of trading the risk-free asset, and \(\frac{\gamma}{O}\) as the proportional cost of trading an option on the underlying stock. Assume that there are \(N_O\) options that trade in secondary markets available for inclusion in the replicating portfolio. Let the symbol \(\cdot LA_{i,j}\) (\(\cdot SA_{i,j}\)) represent the total cash outflow (inflow), after all transaction costs, of going long (short) one share of the stock in state \(\{i,j\}\). The cash outflow (inflow) of going long (short) one share of the risk-free asset is represented by \(\cdot LB_{i,j}\) (\(\cdot SB_{i,j}\)), and the cash outflow (inflow) of going long (short) option \(k\), which costs \(O_{i,j}^k\), is represented by \(\cdot LO_{i,j}\) (\(\cdot SO_{i,j}\)), where each option is written on one share of stock.

In the LP it is assumed that options which are in the replicating portfolio can be traded

only at time zero and at their maturity, time \( T_k \). At these two times the prices of the options are known with certainty. If any options are mis-priced, however, there is no way to know in advance how they will be priced in any binomial state prior to maturity. Therefore, to avoid having to estimate the option’s price at intermediate times, and having to take into account convergence to equilibrium pricing, it is assumed that all option positions are maintained until maturity without revision.

To account for this aspect of trading in the LP, let \( \mu^k_{i,j} \) be an indicator variable which takes on a value of one when option \( k \) is available for trade at time zero and at time \( T_k \), and zero otherwise,

\[
\mu^k_{i,j} = \begin{cases} 
1 & \text{if } i + j = 0 \text{ or } i + j = T_k \\
0 & \text{otherwise}
\end{cases}
\]

By multiplying \( \mu^k_{i,j} \) by the cost of going long or short an option, the LP can be written compactly; when \( \mu^k_{i,j} \) is multiplied by the after-transaction cost option price, the option will not appear in the LP at any time other than time zero and time \( T_k \).

We now define the cash outflow (inflow), net of transactions costs, of going long (short) one share of the stock, one share of the risk-free asset, or one option of type \( k \) after the underlying stock has increased in value \( i \) times and decreased in value \( j \) times:

\[
\begin{align*}
&\cdot L^A_{i,j} = A_{i,j} (1 + \gamma_1) \\
&\cdot S^A_{i,j} = A_{i,j} (1 - \gamma_1) \\
&\cdot L^B_{i,j} = B_{i,j} (1 + \gamma_2) \\
&\cdot S^B_{i,j} = B_{i,j} (1 - \gamma_2) \\
&\cdot L^{O_k}_{i,j} = \mu^k_{i,j} O^{k}_{i,j} (1 + \gamma_3) \\
&\cdot S^{O_k}_{i,j} = \mu^k_{i,j} O^{k}_{i,j} (1 - \gamma_3)
\end{align*}
\]

In the LP, separate decision variables are needed to denote long and short positions in each asset, since linear programming requires that all decision variables be non-negative. The decision variables \( \Phi^{LA}_{i,j} \) and \( \Phi^{SA}_{i,j} \) denote the total number of shares of the risky asset held long (short) in state \( \{i; j\} \). The variables \( \Phi^{LB}_{i,j} \) and \( \Phi^{SB}_{i,j} \) are identical to \( \Phi^{LA}_{i,j} \) and \( \Phi^{SA}_{i,j} \) except that they apply to the risk-free asset. Similarly, the variables \( \Phi^{LO_k}_{i,j} \) and \( \Phi^{SO_k}_{i,j} \) denote the total number of the \( k \)-th option held long (short). In addition to determining the levels
of each asset in each binomial state, the LP must determine the changes in the positions of each asset between binomial states. The decision variables $\Delta_{i,j}^{LA}$ and $\Delta_{i,j}^{SA}$ represent an increase in the long (short) position in the stock in state $\{i;j\}$, $\Delta_{i,j}^{LB}$ and $\Delta_{i,j}^{SB}$ represent an increase in the long (short) position in the risk-free asset, and $\Delta_{i,j}^{LO_k}$ and $\Delta_{i,j}^{SO_k}$ represent an increase in the long (short) position of the $k$-th option. An additional superscript of + or - is used in connection with all variables for all states for which $i + j > 0$ to indicate that the last stock price move was an increase (+) or a decrease (-). For example, $\Delta_{2,1}^{LA+}$ denotes the increase in the long position of the stock in state $\{2;1\}$ where the last move was an up. This implies that the prior state had to have been $\{1;1\}$, not $\{2;0\}$.

In the LP the objective is to minimize the initial cost of the replicating portfolio. The relevant decision variables for establishing the up-front cost of the replicating portfolio are the change variables (Δ). Therefore, the objective function can be stated as:

\[
\text{MINIMIZE} \quad \Delta_{0,0}^{LA} - \Delta_{0,0}^{SA} + \Delta_{0,0}^{LB} - \Delta_{0,0}^{SB} + \sum_{k=1}^{N_O} \left( \Delta_{0,0}^{LO_k} - \Delta_{0,0}^{SO_k} \right) ;
\]

\[
(16)
\]

The first term in the objective function is the cost of establishing a long position in the underlying stock. The second term represents the cash inflow from establishing a short position in the underlying stock. The third and fourth terms are the costs and cash inflows of establishing long and short positions in the risk-free asset, respectively. The terms in the summation represent the cost of establishing positions in the $N_O$ options. The cost of establishing the replicating portfolio is simply the sum of the size of the long (short) position of each security in the portfolio multiplied by the appropriate unit cost (cash inflow) of acquiring the position.

There are three types of constraints which the solution to the LP must satisfy. First, the portfolio must be self-financing, so that when the portfolio is rebalanced, there is enough income from the sale of existing securities held in the portfolio at time $t$ to cover both the purchase of new securities at time $t+1$, and to cover all transaction costs. The following
two constraints ensure that this is the case:

\[
\Delta_{i,j}^{LA+} \cdot LA_{i,j} - \Delta_{i,j}^{SA+} \cdot SA_{i,j} + \Delta_{i,j}^{LB+} \cdot LB_{i,j} - \Delta_{i,j}^{SB+} \cdot SB_{i,j} - \sum_{k=1}^{N_{O}} (\Delta_{i,j}^{LO_{k}} \cdot SO_{k} - \Delta_{i,j}^{SO_{k}} \cdot LO_{k}) + S_{i,j}^{+} = 0
\]  

\[
\Delta_{i,j}^{LA-} \cdot LA_{i,j} - \Delta_{i,j}^{SA-} \cdot SA_{i,j} + \Delta_{i,j}^{LB-} \cdot LB_{i,j} - \Delta_{i,j}^{SB-} \cdot SB_{i,j} - \sum_{k=1}^{N_{O}} (\Delta_{i,j}^{LO_{k}} \cdot SO_{k} - \Delta_{i,j}^{SO_{k}} \cdot LO_{k}) + S_{i,j}^{-} = 0
\]  

The first term in (17) represents the total cost of financing a change in the long position of the underlying stock. Note that terms that are greater than zero represent a cash outflow, while terms less than zero represent cash inflow. The second term represents the cash inflow generated by shorting \(\Delta_{i,j}^{SA+}\) shares of the underlying stock. The third (fourth) term represents the cash outflow (inflow) of increasing the long (short) position in the risk-free asset. The terms inside the summation represent the unwinding of option positions at maturity that were established at time zero. Long options provide income of \(\cdot LO_{k}\) per option, while short options have to be covered at a cost of \(\cdot SO_{k}\) per option. The final term, \(S_{i,j}^{\pm}\), represents the amount by which cash inflow exceeds cost, and converts the inequality to an equality. Constraint (17) applies to the case where the stock increased in value from time \(i + j - 1\) to time \(i + j\). Constraint (18) is identical to (17) except that it pertains to the case where the stock’s value decreased from time \(i + j - 1\) to time \(i + j\). Constraints (17) and (18) must hold for all \(i \leq j\) \(\leq T\) and \(i + j > 0\).

The second set of constraints are referred to as “linking” constraints since they link changes in the positions of each security between consecutive trading times (\(\Delta\)) to the total holdings of that security at one particular time (\(\Phi\)). These constraints simply state that the total number of shares of a security at time \(t = i + j\) must be the total number of shares in the previous state at time \(t = i + j - 1\) plus any changes made to that position at time \(t = i + j\). Linking constraints are not needed for the options, since no changes are made in the option positions after time zero.

The linking constraints for the LP are:

\[
\Phi_{0,0}^{LA} - \Phi_{0,0}^{SA} = \Delta_{0,0}^{LA} - \Delta_{0,0}^{SA} + I_{0,0}^{A}
\]  

\[19\]
The left hand side (LHS) of constraint (19) represents the initial net position of the underlying stock, and the right hand side (RHS) represents the net number of shares purchased. The parameter $I^A_{0,0}$ is the number of shares already owned at time zero, and, therefore, is not a decision variable for the LP. It is necessary to include the number of shares already owned if one uses the LP to optimally rebalance a replicating portfolio when the portfolio already contains some amount of the underlying stock and risk-free bonds.

Constraint (20), which applies to the risk-free asset, and constraint (21), which applies to the $k$-th traded option, parallel constraint (19), which applies to the risky asset. Constraint (22) applies to an upward move in the stock price. The LHS of constraint (22) represents the net position in the underlying stock in state $I_{i;j}$, which must be equal to the number of shares in the previous state, $\Phi^A_{i-1;j} - \Phi^A_{i-1;1}$, plus the net change in position due to trading in state $I_{i;j}$, $\Delta^A_{i;j} - \Delta^A_{i;j}$. Constraint (23) for the risk-free asset parallels constraint (22) for the risky (underlying) stock. Constraints (24) and (25) are identical to (22) and (23) except that they link the total positions in adjacent states when the stock price moves down in price. Constraints (22) through (25) must hold for all $i; j \mid i + j < T$ and $i + j > 0$.

The third constraint that the LP has to satisfy is that the payoffs of the replicating portfolio must equal or exceed the payoffs of the security being replicated (the target). Let $G_{i,j}$ represent the value of the target security in state $I_{i;j}$, where $i + j = T$. Constraint (26) applies to the case where the stock’s value increases from time $i + j - 1$ to time $i + j$. The first (second) term in (26) is the cash flow from liquidating (covering) the total long (short) position in the stock $A$. The third (fourth) term in (26) is the cash flow from liquidating...
(covering) the total long (short) position in the risk-free security $B$. The term inside the summation represents the cash flow from liquidating the $k$-th option. The final term, $S^+_i \cdot j$, represents the amount by which the value of the replicating portfolio exceeds that of the target in state $\{i; j\}$. Constraint (27) is identical to (26), except that it pertains to the case where the stock’s value decreases from time $i + j - 1$ to time $i + j$. Both constraints (26) and (27) must hold for all $i; j \mid i + j = T$.

$$\Phi_{i-1,j}^L \cdot \Phi_{i,j}^S - \Phi_{i-1,j}^S \cdot \Phi_{i,j}^L + \Phi_{i-1,j}^L \cdot \Phi_{i,j}^S - \Phi_{i-1,j}^S \cdot \Phi_{i,j}^L + \sum_{k=1}^{N_O} \left( \Phi_{i,j}^L \Phi_{i,j}^S - \Phi_{i,j}^S \Phi_{i,j}^L \right) - S^+_i \cdot j = G_i \cdot j \tag{26}$$

$$\Phi_{i,j-1}^L \cdot \Phi_{i,j}^S - \Phi_{i,j-1}^S \cdot \Phi_{i,j}^L + \Phi_{i,j-1}^L \cdot \Phi_{i,j}^S - \Phi_{i,j-1}^S \cdot \Phi_{i,j}^L + \sum_{k=1}^{N_O} \left( \Phi_{i,j}^L \Phi_{i,j}^S - \Phi_{i,j}^S \Phi_{i,j}^L \right) - S^-_i \cdot j = G_i \cdot j \tag{27}$$

The objective function, along with constraints (19) through (27), constitute a linear program whose solution provides the lowest cost strategy for replicating the payoffs to an option.
Table 1: Data Used in the Replication Program Example

This table contains the data which are used to generate the results in Table 2. The upper panel specifies the data necessary to construct the binomial lattice. The lower panel contains prices of near-the-money S&P500 index options at 2:00 p.m. on January 4, 1993. The options mature in 46 days.

<table>
<thead>
<tr>
<th></th>
<th>Options</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual Risk-free Rate:</td>
<td>2.97% per year</td>
</tr>
<tr>
<td>Volatility of Index:</td>
<td>10.76% per year</td>
</tr>
<tr>
<td>Index Level:</td>
<td>434.50</td>
</tr>
<tr>
<td>Number of Periods for Binomial Tree:</td>
<td>8</td>
</tr>
<tr>
<td>Time to expiration:</td>
<td>46 days</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Option Number</th>
<th>Bid</th>
<th>Ask</th>
<th>Strike</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.25</td>
<td>10.75</td>
<td>430.00</td>
<td>Call</td>
</tr>
<tr>
<td>2</td>
<td>7.13</td>
<td>7.25</td>
<td>435.00</td>
<td>Call</td>
</tr>
<tr>
<td>3</td>
<td>4.63</td>
<td>5.00</td>
<td>440.00</td>
<td>Call</td>
</tr>
<tr>
<td>4</td>
<td>7.25</td>
<td>7.75</td>
<td>435.00</td>
<td>Put</td>
</tr>
<tr>
<td>5</td>
<td>9.63</td>
<td>10.13</td>
<td>440.00</td>
<td>Put</td>
</tr>
</tbody>
</table>
Table 2: Synthetic Costs and Implied Volatilities as a Function of Strike Price

Long and short prices of synthetic options with various strike prices having 46 days to maturity are computed using the data in Table 1. The LP uses both a dynamic strategy using the stock and bond, and a static strategy using options 1 to 5. Observed bid and ask prices for the corresponding traded options are also shown. Implied volatilities are obtained using the Black-Scholes formula.

<table>
<thead>
<tr>
<th>Target Strike</th>
<th>Long Synthetic Call</th>
<th>Observed Bid</th>
<th>Implied Volatilities (%)</th>
<th>Long Synthetic Call</th>
<th>Observed Bid</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Costs</td>
<td></td>
<td></td>
<td>Costs</td>
<td></td>
</tr>
<tr>
<td>420</td>
<td>18.222</td>
<td>17.63</td>
<td>12.70</td>
<td>11.25</td>
<td></td>
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<tr>
<td>425</td>
<td>14.400</td>
<td>14.00</td>
<td>12.56</td>
<td>11.77</td>
<td></td>
</tr>
<tr>
<td>430</td>
<td>10.781</td>
<td>10.25</td>
<td>12.03</td>
<td>11.11</td>
<td></td>
</tr>
<tr>
<td>435</td>
<td>7.793</td>
<td>7.13</td>
<td>11.79</td>
<td>10.71</td>
<td></td>
</tr>
<tr>
<td>440</td>
<td>5.148</td>
<td>4.63</td>
<td>11.21</td>
<td>10.35</td>
<td></td>
</tr>
<tr>
<td>445</td>
<td>2.870</td>
<td>2.88</td>
<td>10.23</td>
<td>10.25</td>
<td></td>
</tr>
<tr>
<td>450</td>
<td>1.958</td>
<td>1.75</td>
<td>10.80</td>
<td>10.33</td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Target Strike</th>
<th>Short Synthetic Call</th>
<th>Observed Ask</th>
<th>Implied Volatilities (%)</th>
<th>Short Synthetic Call</th>
<th>Observed Ask</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Costs</td>
<td></td>
<td></td>
<td>Costs</td>
<td></td>
</tr>
<tr>
<td>420</td>
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<td>18.38</td>
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<td>13.08</td>
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<td>13.292</td>
<td>14.63</td>
<td>10.32</td>
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<tr>
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<td>9.433</td>
<td>10.75</td>
<td>9.67</td>
<td>11.98</td>
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<tr>
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<td>10.90</td>
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<tr>
<td>440</td>
<td>4.281</td>
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<td>9.77</td>
<td>10.96</td>
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<tr>
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<td>2.010</td>
<td>3.25</td>
<td>8.57</td>
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</tr>
<tr>
<td>450</td>
<td>0.903</td>
<td>2.00</td>
<td>8.18</td>
<td>10.90</td>
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</tbody>
</table>
Table 3: Transaction Costs Needed to Permit Arbitrage - January 1993

This table presents the levels of transaction costs necessary to permit arbitrage for portfolios using cross sections of S&P500 index options formed daily at 1 P.M. during January 1993. Arbitrage is possible when a cash inflow can be produced today with no future liability. The transaction costs presented in the table are the proportional one-way costs on the underlying asset. There are no transaction costs on the risk-free asset and traded options, other than the bid-ask spread.

<table>
<thead>
<tr>
<th>Date of Option Quote</th>
<th>Days to Expiration</th>
<th>Break-Even Transaction Cost</th>
<th>Mean Hedging Error</th>
<th>Standard Deviation of Hedging Error</th>
<th>Ratio of Mean to Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>930104</td>
<td>11</td>
<td>0.0003777</td>
<td>10</td>
<td>5,240</td>
<td>0.00</td>
</tr>
<tr>
<td>930105</td>
<td>10</td>
<td>0.0001236</td>
<td>731</td>
<td>5,181</td>
<td>0.14</td>
</tr>
<tr>
<td>930106</td>
<td>44</td>
<td>0.0000206</td>
<td>322</td>
<td>1,046</td>
<td>0.31</td>
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<tr>
<td>930107</td>
<td>43</td>
<td>0.0008995</td>
<td>77</td>
<td>4,491</td>
<td>0.02</td>
</tr>
<tr>
<td>930108</td>
<td>42</td>
<td>0.0007141</td>
<td>524</td>
<td>4,614</td>
<td>0.11</td>
</tr>
<tr>
<td>930111</td>
<td>39</td>
<td>0.0016479</td>
<td>234</td>
<td>1,792</td>
<td>0.13</td>
</tr>
<tr>
<td>930112</td>
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<td>0.0005630</td>
<td>900</td>
<td>3,584</td>
<td>0.25</td>
</tr>
<tr>
<td>930113</td>
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<td>0.0005630</td>
<td>405</td>
<td>3,433</td>
<td>0.12</td>
</tr>
<tr>
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<td>0.0010506</td>
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<td>2,438</td>
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<tr>
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<td>0.0009612</td>
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<tr>
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<td>0.0007484</td>
<td>143</td>
<td>1,207</td>
<td>0.12</td>
</tr>
<tr>
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<td>0.0010506</td>
<td>272</td>
<td>2,694</td>
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<tr>
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<td>2,273</td>
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<tr>
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<td>2,115</td>
<td>21,503</td>
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<tr>
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<tr>
<td>930126</td>
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<tr>
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<td>0.0004875</td>
<td>435</td>
<td>5,834</td>
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</tr>
<tr>
<td>930128</td>
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<td>0.0001099</td>
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<td>1,081</td>
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</tr>
<tr>
<td>930129</td>
<td>21</td>
<td>0.0000412</td>
<td>151</td>
<td>1,162</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Mean: 0.0005756 374 3,798 0.11
Table 4: Transaction Costs Needed to Permit Arbitrage - January 1995

This table presents the levels of transaction costs necessary to permit arbitrage for portfolios using cross sections of S&P500 index options formed daily at 1 P.M. during January 1995. Arbitrage is possible when a cash inflow can be produced today with no future liability. The transaction costs presented in the table are the proportional one-way costs on the underlying asset. There are no transaction costs on the risk-free asset and traded options, other than the bid-ask spread.

<table>
<thead>
<tr>
<th>Date of Option Quote</th>
<th>Days to Expiration</th>
<th>Break Even Transaction Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>950103</td>
<td>17</td>
<td>0.0006592</td>
</tr>
<tr>
<td>950104</td>
<td>16</td>
<td>0.0008720</td>
</tr>
<tr>
<td>950105</td>
<td>15</td>
<td>0.0011055</td>
</tr>
<tr>
<td>950106</td>
<td>14</td>
<td>0.0007278</td>
</tr>
<tr>
<td>950109</td>
<td>11</td>
<td>0.0006798</td>
</tr>
<tr>
<td>950110</td>
<td>10</td>
<td>0.0000755</td>
</tr>
<tr>
<td>950111</td>
<td>37</td>
<td>0.0019089</td>
</tr>
<tr>
<td>950113</td>
<td>35</td>
<td>0.0005219</td>
</tr>
<tr>
<td>950116</td>
<td>32</td>
<td>0.0001373</td>
</tr>
<tr>
<td>950117</td>
<td>31</td>
<td>0.0004532</td>
</tr>
<tr>
<td>950118</td>
<td>30</td>
<td>0.0003708</td>
</tr>
<tr>
<td>950119</td>
<td>29</td>
<td>0.0005424</td>
</tr>
<tr>
<td>950120</td>
<td>28</td>
<td>0.0006386</td>
</tr>
<tr>
<td>950123</td>
<td>25</td>
<td>0.0007280</td>
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<tr>
<td>950124</td>
<td>24</td>
<td>0.0002609</td>
</tr>
<tr>
<td>950125</td>
<td>23</td>
<td>0.0001991</td>
</tr>
<tr>
<td>950126</td>
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<td>0.0001785</td>
</tr>
<tr>
<td>950127</td>
<td>21</td>
<td>0.0000481</td>
</tr>
<tr>
<td>950130</td>
<td>18</td>
<td>0.0000412</td>
</tr>
<tr>
<td>950131</td>
<td>17</td>
<td>0.0000961</td>
</tr>
</tbody>
</table>

Mean: 0.0005122
Table 5: Initial Replicating Portfolio Positions - January 4, 1993

This table presents the magnitude of the initial positions in the stock, bond, and options on needed to realize the arbitrage opportunity at various levels of transaction costs. A negative sign indicates a short position in that particular security.

<table>
<thead>
<tr>
<th></th>
<th>10% Below At Break-Even Transaction Cost</th>
<th>0% Below At Break-Even Transaction Cost</th>
<th>50% Below At Break-Even Transaction Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Stock:</td>
<td>-7247</td>
<td>-81</td>
<td>-59</td>
</tr>
<tr>
<td>Total Bond:</td>
<td>3146657</td>
<td>-36628</td>
<td>26042</td>
</tr>
<tr>
<td>Option 1 Value:</td>
<td>0</td>
<td>308</td>
<td>22</td>
</tr>
<tr>
<td>Option 2 Value:</td>
<td>4102</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Option 3 Value:</td>
<td>-128263</td>
<td>-4326</td>
<td>-1082</td>
</tr>
<tr>
<td>Option 4 Value:</td>
<td>61426</td>
<td>2607</td>
<td>500</td>
</tr>
<tr>
<td>Option 5 Value:</td>
<td>41</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
This table contains summary statistics for the Monte Carlo simulation results for the twenty trading days in January 1993. For each day, 100 price paths were generated at the break-even transaction cost (BETC) as well as costs of 10% and 50% below this level. The percent negative column represents the percentage of time that the trading strategy lost money (a negative hedging error), and the average dollar loss column is the dollar loss conditional on the hedging error being negative.

<table>
<thead>
<tr>
<th>Transaction Cost Level</th>
<th>Average Transaction Cost ($)</th>
<th>Mean Error ($)</th>
<th>Standard Deviation of Error ($)</th>
<th>Percent Negative if Negative</th>
<th>Average Dollar Loss ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>At BETC</td>
<td>0.0005756</td>
<td>374</td>
<td>3.798</td>
<td>51</td>
<td>-1,629</td>
</tr>
<tr>
<td>10% Below BETC</td>
<td>0.0005181</td>
<td>101</td>
<td>1,394</td>
<td>49</td>
<td>-876</td>
</tr>
<tr>
<td>50% Below BETC</td>
<td>0.0002878</td>
<td>85</td>
<td>623</td>
<td>43</td>
<td>-257</td>
</tr>
</tbody>
</table>
Figure 1: This figure represents the two-period binomial tree on which the LP is based.
Figure 2: This figure shows an example of a three-period binomial model of the stock price evolution. The current price of the asset is denoted by $A_{0,0}$. The future value of the stock price after $i$ ups and $j$ downs have occurred is denoted as $A_{i,j}$. 