Exchange Rate Volatility
and the
Forward Premium Anomaly

Jeremy J. Graveline*

JOB MARKET PAPER

December 12, 2005

Abstract

Existing research has yet to identify a risk premium that accounts for important empirical properties of exchange rate returns such as the forward premium anomaly: the tendency for currencies with high interest rates to appreciate against currencies with lower interest rates. This paper examines the forward premium anomaly through the lens of an arbitrage-free pricing model for the exchange rate and term structure of interest rates in two currencies. Previous papers in this literature have failed to match exchange rate volatility, which is a vital component of the risk premium in exchange rate returns. The model in this paper generalizes previous models and is estimated using the joint time-series of U.S. and U.K. swap rates, dollar/pound exchange rate returns, and prices of at-the-money exchange rate options. I include option prices because they are highly sensitive to the level of volatility and to the pricing of volatility risk. When options are used to estimate the model, it successfully captures both exchange rate volatility and the term structure of interest rates in the U.S. and U.K. Using simulated data, I show that the model also replicates the empirical findings in Fama (1984) and accounts for the forward premium anomaly.

*Stanford Graduate School of Business, email: graveline jeremy@gsb.stanford.edu

I am grateful for helpful comments from Greg Bauer, David Bolder, Peter DeMarzo, Antonio Diez de los Rios, Darrell Duffie, Steve Grenadier, Scott Joslin, Camelia Kuhnen, Matthew McBrady, Stefan Nagel, Peter Reiss, Ken Singleton, and Ilya Strebulaev. The most recent version of this paper can be downloaded from my website http://www.stanford.edu/~jjgravel/.
1 Introduction

Changes in exchange rates are a significant determinant of returns on foreign investments, yet existing research has failed to identify a risk premium that reconciles the empirical properties of exchange rate returns with prices of other assets in financial markets. One such empirical property that has eluded models of the risk premium in exchange rate returns is the forward premium anomaly: the tendency for currencies with high interest rates to appreciate against currencies with lower interest rates, rather than depreciate as uncovered interest rate parity would suggest. This paper aims to produce an empirical model that successfully captures exchange rate volatility and the term structure of interest rates in both currencies, and examine whether the exchange rate risk premium inherent in the model also accounts for the empirical properties of the forward premium anomaly documented in Fama (1984).

To address this question, I present and estimate a dynamic arbitrage-free empirical pricing model for the exchange rate and term structure of interest rates in two currencies. The risks that constitute exchange rate volatility are a critical element of the risk premium in exchange rate returns. In order to measure this risk premium well, it is useful to add a financial security not used in prior work on the forward premium anomaly, exchange rate options, whose price is highly sensitive to the level of volatility and to the pricing of volatility risk. I find that options provide valuable information about exchange rate volatility that is much harder to identify using only time-series data on exchange rates and interest rates in each currency.

The same factors that determine the risk premium in exchange rate returns can also affect risk premia in the term structure of interest rates in each currency. Moreover, exchange rates and nominal interest rates may both depend on expectations about the same macroeconomic variables (e.g. inflation). Hence, it is logical to examine the risk premium in exchange rate returns jointly with interest rates. Nielsen and Saá-Requejo (1993), Backus et al. (2001), Bansal (1997), and Hodrick and Vassalou (2002) first studied the forward premium anomaly in the context of two-currency term structure models. More recently, Ahn (2004), Dewachter and Maes (2001), and Incu and Lu (2004) have used more flexible models. These previous papers focused primarily on capturing interest rates in both currencies, but failed to match (or neglected to examine) exchange rate volatility. Since the risks that comprise exchange rate volatility are a vital component of the risk premium in exchange rate returns, a model that does not match volatility cannot convincingly account for the forward premium anomaly.

---

1 In the most flexible models it has traditionally been too expensive to compute option prices and include them in model estimation. This paper uses a cumulant expansion technique to efficiently compute option prices and facilitate estimation. This cumulant expansion technique was first applied to option pricing in an affine framework by Almeida, Graveline, and Joslin (2005).

2 A notable exception is Brandt and Santa-Clara (2002) who use an additional risk factor to match exchange rate volatility. However, they do not examine the forward premium anomaly because they assume that the risk factor is not priced and therefore does not affect the risk premium in exchange rate returns.
The empirical model in this paper extends the most flexible class of single-currency affine term structure models to a two-currency setting by *jointly* specifying the dynamics of the nominal exchange rate and the minimum variance nominal pricing kernel (stochastic discount factor).\(^3\) The model generalizes previous two-currency affine term structure models that have been empirically studied in the literature by allowing the risk factors to affect the market prices of risk directly, and not only through factor volatilities. Duffee (2002), Dai and Singleton (2002), and Cheridito et al. (2005) find that this specification for the market prices of risk is required to match the empirical properties of risk premia in the term structure of interest rates within an affine model. Brennan and Xia (2005) also show theoretically that two-currency affine term structure models with this feature have additional flexibility for capturing the forward premium anomaly.\(^4\)

I estimate a 4-factor version of the model with quasi-maximum likelihood using the joint time-series of U.S. and U.K. swap rates, dollar/pound exchange rate returns, and at-the-money option prices on the exchange rate. The model nests previous two-currency term structure models that have been empirically studied in the literature, yet it does not accurately capture exchange rate volatility unless options are included in estimation. The sample annualized exchange rate volatility in the data calculated using one-week returns is 7.91%, but when the model is estimated without using options, the corresponding mean exchange rate volatility implied by the model is 13.87%. Moreover, the mean absolute relative pricing error for 6-month at-the-money options is 54.39%. This result is consistent with Dewachter and Maes (2001) who also find that their model (which is estimated without using options) overstates exchange rate volatility. By comparison, when the model is estimated using options, the fit to exchange rate volatility improves dramatically. The mean exchange rate volatility implied by the model is 9.65% and the mean absolute relative pricing error for 6-month at-the-money options is only 7.89%.

When options are included in estimation, the number of factors in the model remains the same but the fit to exchange rate volatility improves dramatically. Since the factors that affect exchange rate volatility also affect interest rates, a natural question is whether the nature of the risk factors changes when options are included and whether, as a result, there is a deterioration in the model’s ability to fit the term structures of interest rates in the U.S. and U.K. I explore these issues in depth and find that when options are included in estimation there is a parallel shift in the level of the risk factors, but only the parameters

---

\(^3\)See Duffie and Kan (1996) for a description of affine term structure models. Dai and Singleton (2000), Duffee (2002), and Cheridito et al. (2005) discuss completely, essentially, and extended affine specifications for the market prices of risk respectively. This paper uses an extended affine market price of risk which is the most general of these three.

\(^4\)While the theoretical analysis in Brennan and Xia (2005) focuses on two-currency models, the empirical analysis instead uses regression analysis to examine the relationship between exchange rate returns and the pricing kernels they estimate in different currencies using single-currency term structure models. Two-currency (international) pricing models are a significant extension of these single-currency models as they directly specify the joint relationship between the term structure of interest rates in both currencies, and between interest rates and exchange rate returns.
governing exchange rate volatility change significantly. Moreover, there is little change in the (very good) fit to interest rates.

Having produced an empirical model that successfully captures exchange rate volatility and the term structure of interest rates in the U.S. and U.K., the paper then examines whether the model also successfully accounts for the forward premium anomaly. The forward premium anomaly is characterized by the following regression from Fama (1984),

\[
\ln S_{t+\Delta t} - \ln S_t = \alpha + \beta (r_t - r_f^t) + \varepsilon_t,
\]

where \(S_t\) is the nominal exchange rate (expressed in units of domestic currency per unit of forward currency), \(r_t\) is the nominal domestic interest rate until \(t+\Delta t\), and \(r_f^t\) is the nominal foreign interest rate until \(t+\Delta t\). Uncovered interest rate parity predicts that \(\hat{\beta} = 1\), but Fama (1984) and others find that \(\hat{\beta} < 0\) for most currency pairs. Fama (1984) shows that a negative regression coefficient \(\hat{\beta}\) implies that the risk premium in exchange rate returns is negatively correlated with, and more variable than, the difference in interest rates between the two currencies. In the weekly data set used in this paper (August 2001 to July 2005), \(\hat{\beta} = -4.50\) with a 95% confidence interval of \([-13.15, 4.15]\). Using the historical quasi-maximum likelihood estimates, I simulate four years of weekly exchange rate and interest rate data 1,000 times. The mean regression coefficient \(\hat{\beta}\) from the simulations is \(-5.11\) and the corresponding 95% confidence interval is \([-16.27, 6.41]\). Thus, the model replicates Fama’s empirical findings and successfully accounts for the forward premium anomaly.

The empirical properties of the model contradict those of previous studies. The models in Brandt and Santa-Clara (2002) and Dewachter and Maes (2001) both impose the assumption that there is an unpriced risk factor that affects exchange rate volatility but not exchange rate returns or interest rates in either currency. I also allow for such a risk factor, but do not assume that it is unpriced. The model estimates imply that all risk factors are priced and none of them affect only exchange rates. In addition, previous two-currency term structure models have attempted to explain the forward premium anomaly by imposing a distinction between local factors that affect interest rates in only one currency, and global factors that affect interest rates in both currencies. Instead, this paper allows each latent risk factor to be either local or global in nature, and I find that all risk factors affect interest rates in both the U.S. and U.K., although they do have asymmetric effects on interest rates in each currency.

The empirical results also relate to recent papers by Bakshi et al. (2005) and Brandt et al. (2005). Bakshi et al. (2005) use option prices to estimate a dynamic exchange rate model with jumps, but do not examine the forward premium anomaly because they assume that the term structure of interest rates in both currencies is constant and flat. To justify this assumption, they assert that the risks that affect exchange rate returns are independent of those that affect interest rates. I also use options and match exchange rate volatility,

\[5\]For example, see Backus et al. (2001), Ahn (2004), and Dewachter and Maes (2001).
but instead I find that the same risks affect both exchange rates and the term structure of interest rates. Brandt et al. (2005) argue that exchange rate volatility is smaller than the volatility of the pricing kernels in the relevant currencies, which in turn implies that these pricing kernels are highly correlated. The model I estimate in this paper also matches these empirical properties.

The remainder of the paper is organized as follows. Section 2 describes dynamic international asset pricing models and the forward premium anomaly. Section 3 presents the specific class of two-currency affine term structure models I study in the paper and discusses valuation of zero coupon bonds and exchange rate options. Section 4 describes the data and estimation procedure. Section 5 discusses the empirical results and Section 6 concludes. Figures and most technical details are relegated to the appendix.

2 Dynamic International Asset Pricing Models

In the absence of arbitrage, it is well known that there exists a unique minimum variance pricing kernel (stochastic discount factor) $M_t$ such that the price of any payoff $P_T$ is given by

$$P_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} P_T \right].$$

(1)

A dynamic asset pricing model specifies the dynamics of the minimum variance pricing kernel. If the pricing kernel $M$ follows a diffusion process of the form

$$dM_t = -M_t r_t dt - M_t \Lambda_t^\top dW_t,$$

and the price $P$ of an asset also follows a diffusion process that depends on the Brownian motion $W$ with volatility $\sigma_t$, then

$$dP_t = P_t \left[ r_t + \sigma_t \Lambda_t \right] dt + P_t \sigma_t dW_t.$$  

(2)

The drift $r_t$ of the pricing kernel is commonly referred to as the short interest rate and the volatility $\Lambda_t$ is commonly referred to as the market price of risk. With these labels, the risk

---

6See Duffie (2001) for a formal treatment of pricing kernels. For intuition, the pricing kernel $M$ can be thought of as the marginal utility $U'$ of the representative investor, in which case

$$P_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} P_T \right] = \mathbb{E}_t \left[ \frac{U'}{U_t'} P_T \right].$$

5For ease of exposition, I discuss a continuous-time diffusion framework. The discrete-time counterpart has identical economic implications with slightly more notational complexity. The diffusion framework can be generalized to include jumps at the expense of some additional complexity in model estimation.
premium (or expected return in excess of the short interest rate $r_t$) is $\sigma_t \Lambda_t$, which can be thought of as the amount of risks contained in the vector $\sigma_t$, weighted by the market prices $\Lambda_t$ of those risks.

In an international asset pricing model, if financial markets are open and integrated then the pricing kernel $M$ must also price payoffs in a foreign currency when they are exchanged to the domestic currency. Let $P_t^f$ be the price in foreign currency of a payoff $\tilde{P}_t^f$ in foreign currency. If $S$ is the exchange rate (expressed in units of domestic currency per unit of foreign currency) and $M$ is an international pricing kernel, then it must also be the case that

$$S_t \tilde{P}_t^f = E_t \left[ \frac{M_T}{M_t} \left( S_T \tilde{P}_T^f \right) \right].$$

A dynamic international asset pricing model must also specify the dynamics of the exchange rate. Let $r_t^f$ be the short interest rate in the foreign currency and suppose that the exchange rate also follows a diffusion process

$$dS_t = S_t \left[ r_t - r_t^f + \sigma_t^S \Lambda_t \right] dt + S_t \sigma_t^S dW_t.$$  

If the foreign asset’s price $P_t^f$ in the foreign currency depends on $W$ with volatility $\sigma_t^f$, then

$$d\tilde{P}_t^f = \tilde{P}_t^f \left[ r_t^f + \sigma_t^f \left( \Lambda_t^\top - \sigma_t^S \right)^\top \right] dt + \tilde{P}_t^f \sigma_t^f dW_t.$$  

From equation (4), the expected change in the exchange rate is equal to the difference $r_t - r_t^f$ between the domestic and foreign short interest rate, plus a risk premium $\sigma_t^S \Lambda_t$. In a risk-neutral world with $\Lambda_t = 0$, the uncovered interest rate parity hypothesis says that the expected change in the exchange rate is equal to the difference $r_t - r_t^f$ between the domestic and foreign short interest rates. Empirically, exchange rates do not tend to change by the

---

8Previous papers in this literature have characterized international asset pricing models using domestic and foreign pricing kernels $M$ and $M^f$. From equations (1) and (3), it is easy to verify that

$$\frac{M_t^f}{M_t} := \frac{M_T}{M_t} \frac{S_T}{S_t},$$

is an international pricing kernel that prices assets in the foreign currency when payoffs are denominated in either the domestic or foreign currency. Section C in the appendix shows that these two approaches are equivalent, and if $M$ is the minimum variance pricing kernel denominated in the domestic currency, then $M^f$ is in fact the minimum variance pricing kernel denominated in the foreign currency.

9This characterization of the exchange rate in terms of the foreign short interest rate $r_t^f$ is without loss of generality. If the exchange rate follows a diffusion process of the form

$$dS_t = S_t \kappa_t^S dt + S_t \sigma_t^S dW_t,$$

then it is easy to show that $r_t^f := r_t + \sigma_t^S \Lambda_t - \kappa_t^S$ is the short interest rate in the foreign currency.

---

6
difference in interest rates, as uncovered interest rate parity would suggest. We know that exchange rates are volatile (i.e. $\sigma_t^S \neq 0$) and there is strong evidence that investors are not risk averse (i.e. $\Lambda_t \neq 0$). Hence, it is not particularly puzzling that exchange rate returns contain a risk premium (i.e. $\sigma_t^S \Lambda_t \neq 0$).

What has proven puzzling, and is often referred to as the forward premium anomaly, is that currencies with high interest rates actually tend to appreciate against currencies with lower interest rates, rather than depreciate. More formally, Fama (1984) performs the following regression

$$\ln S_{t+\Delta t} - \ln S_t = \alpha + \beta \left( r_t - r_t^f \right) + \varepsilon_t, \quad (6)$$

and finds that $\hat{\beta} < 0$ for most currency pairs. Using the exchange rate dynamics in equation (4), the population value of $\beta$ in equation (6) is

$$\beta = \frac{\text{cov} \left( \ln S_{t+\Delta t} - \ln S_t, r_t - r_t^f \right)}{\text{var} \left( r_t - r_t^f \right)} \approx \frac{\text{cov} \left( r_t - r_t^f + \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^{S\top}, r_t - r_t^f \right)}{\text{var} \left( r_t - r_t^f \right)} \quad (10)$$

Fama shows that $\beta < 0$ implies that

$$\text{var} \left( r_t - r_t^f \right) < - \text{cov} \left( \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^{S\top}, r_t - r_t^f \right) < \text{var} \left( \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^{S\top} \right) \quad (11)$$

That is, the implied risk premium in exchange rate returns is negatively correlated with, and

10There is a convexity adjustment to the drift when the dynamics of the exchange rate are expressed in logs, so that from equation (4),

$$d \ln S_t = \left[ r_t - r_t^f + \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^{S\top} \right] dt + \sigma_t^S d W_t,$$

which implies that for small $\Delta t$,

$$\ln S_{t+\Delta t} - \ln S_t \approx \left[ r_t - r_t^f + \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^{S\top} \right] \Delta t + \sigma_t^S \left( W_{t+\Delta t} - W_t \right).$$
more variable than the difference in interest rates.

One could easily specify an empirical international pricing model with an exchange rate risk premium and short interest rates in both currencies that satisfy the conditions in Fama. However, the short interest rates and components of the risk premium do not only affect exchange rate returns, they also affect other asset prices and, in particular, the term structures of interest rates in both currencies. Recall from equation (2) that the domestic short interest rate \( r_t \) and market prices of risk \( \Lambda_t \) determine the term structure of interest rates in the domestic currency. Similarly, equation (5) illustrates that the term structure of interest rates in the foreign currency depends on the foreign short interest rate \( r_f^t \) and the market prices of risk minus exchange rate volatility \( \Lambda_t - \sigma_S^t \). Indeed, Fama’s finding is often referred to as an anomaly because existing research has struggled to produce an empirical international asset pricing model that captures the term structures of interest rates in both currencies, with an inherent exchange rate risk premium that also satisfies the conditions in equation (7) characterized in Fama (1984).

For an international asset pricing model to be considered as a viable candidate for satisfying the forward premium anomaly, it must also successfully capture exchange rate volatility, yet this additional criterion has been neglected in previous literature. Since the risks that comprise exchange rate volatility \( \sigma_t^S \) are a vital component of the risk premium \( \sigma_t^S \Lambda_t \) in exchange rate returns, a model that does not match volatility cannot convincingly account for the forward premium anomaly. This paper aims to produce an empirical model that successfully captures exchange rate volatility and the term structure of interest rates in both currencies, and test whether that model also matches the empirical properties of the implied risk premium in exchange rate returns documented in Fama (1984).

In addition to being relevant for the forward premium anomaly, exchange rate volatility \( \sigma_t^S \) is also particularly relevant for the returns on investments in assets with payoffs denominated in a foreign currency. From (5), if a foreign asset’s price \( \hat{P}^f_t \) in the foreign currency depends on \( W \) with volatility \( \sigma_t^f \), then the dynamics of its price \( P_t^f := S_t \hat{P}^f_t \) in the domestic currency

\[ 0 < \text{cov} \left( r_t - r_f^t + \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^S^\top, r_t - r_f^t \right), \]

\[ \downarrow \]

\[ \text{var} \left( r_t - r_f^t \right) < -\text{cov} \left( \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^S^\top, r_t - r_f^t \right), \]

\[ \downarrow \]

\[ \text{var} \left( r_t - r_f^t + \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^S^\top \right) < \text{var} \left( \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^S^\top \right) + \text{cov} \left( \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^S^\top, r_t - r_f^t \right), \]

\[ \downarrow \]

\[ -\text{cov} \left( \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^S^\top, r_t - r_f^t \right) < \text{var} \left( \sigma_t^S \Lambda_t - \frac{1}{2} \sigma_t^S \sigma_t^S^\top \right). \]
are given by
\[
dP^f_t = P^f_t \left[ r_t + \left( \sigma_t^f + \sigma_t^S \right) \Lambda_t \right] dt + P^f_t \left( \sigma_t^f + \sigma_t^S \right) dW_t.
\]

Thus, exchange rate volatility affects both the volatility \( \sigma_t^f + \sigma_t^S \) and expected excess return \( \left( \sigma_t^f + \sigma_t^S \right) \Lambda_t \) of foreign assets when their price is exchanged to the domestic currency.

Before proceeding to discuss the specific empirical model I study in the paper, there are two special cases of this framework that serve as important illustrative examples. First, there is a significant distinction between the minimum variance pricing kernel that prices only domestic assets and that which prices both domestic and foreign assets. For example, suppose that domestic asset prices do not depend on the \( i \)th risk factor (i.e. \( [\sigma_t]_i = 0 \)). In this case, the minimum variance pricing kernel that prices only domestic assets will not depend on the \( i \)th risk factor since that factor does not affect domestic asset prices. However, if the exchange rate or a foreign asset price does depend on the \( i \)th risk factor (i.e. \( [\sigma_t^S]_i \neq 0 \) or \( [\sigma_t^f]_i \neq 0 \)), then the minimum variance international pricing kernel that prices both domestic and foreign assets will depend on that factor if it is priced. Brennan and Xia (2005) study the empirical relationship between the exchange rate and single-currency pricing kernels but, as this example illustrates, single-currency pricing kernels can be very different from their counterparts that price assets in two currencies. Second, the exchange rate can depend on factors that do not affect the risk premium. For example, if the exchange rate depends on the \( i \)th factor (i.e. \( [\sigma_t^S]_i \neq 0 \) but that risk factor is not priced (i.e. \( [\Lambda_t]_i = 0 \)), then the risk premium \( \sigma_t^S \Lambda_t \) will not depend on that factor (since \( [\sigma_t^S]_i \), \( [\Lambda_t]_i = 0 \)). Brandt and Santa-Clara (2002) and Dewachter and Maes (2001) both estimate models in which they impose this restriction, but instead, this paper allows all risks in the model to be priced.

## 3 Model and Valuation

This section presents the specific class of two-currency affine term structure models that I study in the paper and discusses the pricing of zero coupon bonds and exchange rate options.

Following the large literature on affine dynamic asset pricing models, I model the minimum variance nominal pricing kernel as a diffusion process
\[
dM_t = -M_t r_t dt - M_t \Lambda_t^\top dW_t,
\]
where
\[
\begin{align*}
r_t & := \rho_0 + \rho_1 \cdot X_t, \quad (8b) \\
\Lambda_t & := \left( \sqrt{\Delta [\alpha + \beta X_t]} \right)^{-1} \left[ (\mathcal{K}_0^p - \mathcal{K}_0) + (\mathcal{K}_1^p - \mathcal{K}_1) X_t \right], \quad (8c)
\end{align*}
\]
and

\[ dX_t = \left[ K_0 + K_1 X_t \right] dt + \sqrt{\Delta \left[ \alpha + \beta X_t \right]} \, dW_t \quad (8d) \]

I have used the notation \( \Delta \cdot \) to denote a square matrix with its vector argument along the diagonal. Dai and Singleton (2000) and Cheridito et al. (2005) provide parameter restrictions so that the process \( X \) is admissable and the model parameters are identifiable.

To extend this model to a two-currency setting, I also model the exchange rate as an affine process

\[ dS_t = S_t \left[ r_t - r^f_t + \Sigma \sqrt{\Delta \left[ \alpha + \beta X_t \right]} \Lambda_t \right] dt + S_t \Sigma \sqrt{\Delta \left[ \alpha + \beta X_t \right]} \, dW_t, \quad (8e) \]

where

\[ r^f_t := \rho^f_0 + \rho^f_1 \cdot X_t. \quad (8f) \]

As noted in the introduction, the model characterized by equation (8) generalizes previous two-currency affine term structure models that have been empirically studied in the literature. Previous papers have instead modelled the market prices of risk as

\[ \Lambda_t := \sqrt{\Delta \left[ \alpha + \beta X_t \right]} \Lambda_0. \quad (9) \]

That is, the market prices of risk in equation (9) are simply constant multiples of the volatility of the factors in equation (8d). By contrast, when the market prices of risk are modelled according to (8c), the risk factors can affect the market prices of risk directly, and not only through factor volatilities.

The specification in equation (9) for the market prices of risk is referred to in the term structure literature as completely affine, and the generalization in equation (8e) is referred to as extended affine. Duffee (2002), Dai and Singleton (2002), and Cheridito et al. (2005) find that this extension is required to match the empirical properties of risk premia in the term structure of interest rates within an affine model. The extended affine specification for the market prices of risk also provides additional flexibility for capturing the forward premium

---

12The dynamics of the state vector \( X \) under the martingale pricing measure \( Q \) defined by

\[ \frac{dQ}{dp} \bigg|_t := e^{\int_0^t r_u \, du} \frac{M_t}{M_0} = e^{-\frac{1}{2} \int_0^t \Lambda_u^\top \Lambda_u \, du - \int_0^t \Lambda_u^\top dW_u}, \]

are

\[ dX_t = \left[ K_0 + K_1 X_t \right] dt + \sqrt{\Delta \left[ \alpha + \beta X_t \right]} \, dW^Q_t, \]

where

\[ W^Q_t := W_t + \int_0^t \Lambda_u \, du, \]

is a Brownian motion under \( Q \).
anomaly. Recall from equation (7) that the forward premium anomaly pertains to the risk premium $\sigma_t^S \Lambda_t$ in exchange rate returns. When the market prices of risk are completely affine, this risk premium is

$$\sigma_t^S \Lambda_t = \Sigma \Delta [\alpha + \beta X_t] \Lambda_0.$$  \tag{10}

Instead, with an extended affine specification for the market prices of risk this risk premium is

$$\sigma_t^S \Lambda_t = \Sigma \left[ (K^S_0 - K_0) + (K^S_1 - K_1) X_t \right].$$  \tag{11}

The exchange rate risk premium in equation (11) provides more flexibility for matching the forward premium anomaly than its counterpart in equation (10).

Duffie and Kan (1996) show that in the affine framework characterized by equation (8), domestic zero coupon bond prices are given by

$$\mathbb{E}_t \left[ \frac{M_T}{M_t} \right] = e^{A(T-t)+B(T-t)\cdot X_t},$$

where $A$ and $B$ satisfy Riccati ODEs (expressed in integral form)

$$B(\tau) = \int_0^\tau -\rho_1 + K_1^T B(u) + \frac{1}{2} \beta^T \Delta [B(u)] B(u) \, du,$$

$$A(\tau) = \int_0^\tau -\rho_0 + K_0^T B(u) + \frac{1}{2} \alpha^T \Delta [B(u)] B(u) \, du.$$

Similarly, foreign zero coupon bond prices are

$$\mathbb{E}_t \left[ \frac{M_T S_T}{M_t S_t} \right] = e^{A^f(T-t)+B^f(T-t)\cdot X_t},$$

where $A^f$ and $B^f$ also satisfy Riccati ODEs

$$B^f(\tau) = \int_0^\tau -\rho_1^f + [\beta^T \Delta [\Sigma] + K_1^T] B^f(u) + \frac{1}{2} \beta^T \Delta [B^f(u)] B^f(u) \, du,$$

$$A^f(\tau) = \int_0^\tau -\rho_0^f + [\alpha^T \Delta [\Sigma] + K_0^T] B^f(u) + \frac{1}{2} \alpha^T \Delta [B^f(u)] B^f(u) \, du.$$

Solutions to the Riccati ODEs can be efficiently computed numerically, therefore domestic and foreign zero coupon bond prices can be easily used to estimate the model.

Using Itô’s Lemma, it is straightforward to verify that for $P_t(T) := e^{A(T-t)+B(T-t)\cdot X_t}$,

$$dP_t(T) = P_t(T) \left\{ \sigma_t (T) \Lambda_t \left[ r_t + B(T-t)^T \left[ (K^S_0 - K_0) + (K^S_1 - K_1) X_t \right] \right] \, dt + \sigma_t (T) B(T-t)^T \sqrt{\Delta [\alpha + \beta X_t]} \, dW_t \right\},$$

which is of the form in equation (2).
This paper differs from previous literature in that I also include exchange rate option prices in model estimation. Exchange rate option prices (in domestic currency) are given by

\[ E_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] = E_t \left[ \frac{M_T}{M_t} S_T 1_{(S_T \geq K)} \right] - K E_t \left[ \frac{M_T}{M_t} 1_{(S_T \geq K)} \right]. \]

In general, this expectation is expensive to compute (for example, using Monte Carlo simulation), which prohibits the inclusion of option prices in model estimation. Duffie et al. (2000) show that for affine pricing models, exchange rate option prices can be computed (using transform analysis and the Lévy inversion formula) as

\[ E_t \left[ \frac{M_T}{M_t} S_T^b 1_{(S_T \geq K)} \right] = \frac{1}{2} E_t \left[ \frac{M_T}{M_t} S_T^b \right] - \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{v} \Im \left\{ \frac{1}{v} E_t \left[ \frac{M_T}{M_t} S_T^{b-iw} \right] \right\} dv, \tag{12} \]

where

\[ E_t \left[ \frac{M_T}{M_t} S_T^b \right] = e^{A^S(\delta, T-t)+B^S(\delta, T-t)\cdot X_t} S_t^b, \]

where \( A^S \) and \( B^S \) satisfy Riccati ODEs given in Appendix C.

If the solutions \( A^S \) and \( B^S \) to these Riccati ODEs are known in closed form, then exchange rate options can be computed as a simple one-dimensional integral given in equation (12). However, in the most flexible affine models, \( A^S \) and \( B^S \) are not known in closed form and must also be computed numerically. Thus, this method for computing option prices is still prohibitively expensive. Instead, this paper uses a cumulant expansion technique which essentially uses a Taylor series to expand the integrand in equation (12). Using the cumulant expansion, exchange rate option prices can be computed without numerical integration (aside from solving the Riccati ODEs) using only the density and cumulative distribution of the Normal distribution (for which there exist excellent closed-form approximations). This technique was first introduced to option pricing by Jarrow and Rudd (1982) and was developed by Almedia, Graveline, and Joslin (2005) for the general class of affine models (including the model in this paper). For completeness, Appendix C describes the details of this method applied to pricing exchange rate options.

4 Data and Estimation Procedure

I use a four-factor\(^{14} \)
version of the general model described in the previous section to empirically examine the joint dynamics of the exchange rate and interest rates in the United

\(^{14}\)Three factors are commonly used to model the U.S. yield curve. Adding an additional factor to also capture the U.K. yield curve and exchange rate dynamics is a rather parsimonious extension to an international setting.
States and the United Kingdom. Specifically, the empirical model I estimate is

\[ dX_t = [K^p_0 + K^p_1 X_t] \, dt + \sqrt{\Delta} [\alpha + \beta X_t] \, dW_t, \]

\[ dM_t = -M_t r_t \, dt - M_t \Lambda_t^\top \, dW_t, \]

\[ r_t := \rho_0 + \rho_1 \cdot X_t, \]

\[ \Lambda_t := (\sqrt{\Delta} [\alpha + \beta X_t])^{-1} \left([K^p_0 - K_0] + (K^p_1 - K_1) X_t\right), \]

\[ dS_t = S_t \left[r_t - r_f^l + \Sigma \sqrt{\Delta} [\alpha + \beta X_t] \Lambda_t\right] \, dt + S_t \Sigma \sqrt{\Delta} [\alpha + \beta X_t] \, dW_t, \]

\[ r_f^l := \rho^l_0 + \rho^l_1 \cdot X_t, \]

where

\[ K^p_0 := \begin{bmatrix} 0 & 0 & K^p_{03} & K^p_{04} \end{bmatrix}^\top, \]

\[ K_0 := \begin{bmatrix} K_{01} & K_{02} & K_{03} & K_{04} \end{bmatrix}^\top, \]

\[ K^p_1 := \begin{bmatrix} K^p_{111} & 0 & K^p_{113} & K^p_{114} \\ K^p_{121} & K^p_{122} & K^p_{123} & K^p_{124} \\ 0 & 0 & K^p_{133} & 0 \\ 0 & 0 & K^p_{143} & K^p_{144} \end{bmatrix}, \]

\[ K_1 := \begin{bmatrix} K_{111} & 0 & K_{113} & K_{114} \\ K_{121} & K_{122} & K_{123} & K_{124} \\ 0 & 0 & K_{133} & 0 \\ 0 & 0 & K_{143} & K_{144} \end{bmatrix}, \]

\[ \alpha + \beta X_t := \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta_{13} & \beta_{14} \\ 0 & \beta_{23} & \beta_{24} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}^\top, \]

\[ \rho_1 := \begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \end{bmatrix}^\top, \]

\[ \rho_f^l := \begin{bmatrix} \rho_f_{11} & \rho_f_{12} & \rho_f_{13} & \rho_f_{14} \end{bmatrix}^\top, \]

\[ \Sigma := \begin{bmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & \Sigma_4 \end{bmatrix}. \]

This particular empirical specification has two factors following Feller, or CIR, processes with stochastic volatility and two factors following essentially Gaussian processes. In the model, interest rate volatility, exchange rate volatility, and market prices of risk are vary stochastically. Note that this model does not impose a distinction between local risk factors that affect prices and returns in only one currency and global or common risk factors that affect prices and returns in both currencies. Previous papers in this literature (such

\[ ^{15} \text{In a single-currency setting, the model is similar to the } A_2 (4) \text{ model described in } \text{Dai and Singleton} (2000). \]
as Backus et al. (2001), Ahn (2004), and Dewachter and Maes (2001) have imposed this distinction. Instead, this paper allows the data to determine whether each risk factor is local or global in nature.

To estimate the model, I use weekly data on U.S. and U.K. Libor and swap rates, changes in the dollar/pound exchange rates, and option-implied volatilities from at-the-money exchange rate options. The data was obtained from Datastream and spans the time period August 15, 2001 to July 6, 2005 (the period for which option implied volatilities are available). Spot exchange rates and the Libor rates in each currency were used to convert the implied volatilities to option prices using Black’s formula. Swap rates were converted to continuously compounded zero coupon rates by bootstrapping the swap zero curve under the assumption that forward rates are constant between observed maturities. The spot dollar/pound exchange rate over the sample period is plotted in Figure 1 and the 3-month and 5-year U.S. and U.K. zero coupon swap rates are plotted in Figure 2. Over the sample period, interest rates in the U.K. were higher than their counterparts in the U.S., yet the pound appreciated against the dollar.

The paper uses quasi-maximum likelihood\(^\text{16}\) to estimate two versions of this model. In both versions I assume that the 3-month U.S. Libor rate, 5-year U.S. zero coupon swap rate, 3-month U.K. Libor rate, and the change in the dollar/pound exchange rate are priced correctly by the model so that I can invert these quantities at each time period to obtain the

\(^{16}\)As shown in Duffie (2001), the exact mean and variance of an affine process are known in closed form.
implied latent states. The choice of which prices are used to invert for the implied states is admittedly somewhat arbitrary. The forward premium anomaly pertains to short-dated interest rates, therefore I choose to exactly match a short-dated interest rate in both the U.S. and U.K. By assuming that changes in the exchange rate are priced exactly, I allow for a risk-factor in the model that does not have to affect interest rates in either the U.S. and U.K. Both Brandt and Santa-Clara (2002) and Dewachter and Maes (2001) estimate models with risk factors that affect exchange rates but not interest rates. For the remaining price that is matched exactly in the model, I use a long-dated U.S. interest rate that is sensitive to the market price of risk.

For both estimates of the model, I assume that the following quantities are priced with error:

- 1-month and 6-month U.S. and U.K. Libor rates;

---

17 This estimation technique was introduced by Chen and Scott (1993) and is widely used when estimating dynamic asset pricing models with latent states. Zero coupon swap rates are affine functions of the states and therefore can be easily inverted to obtain the implied states. The change in the log of the exchange rate can be written as

$$\ln S_{t+\Delta t} - \ln S_t = (K_0^S - K_1^S K_1^P)^\Delta t + \Sigma (X_{t+\Delta t} - X_t) + (K_1^P K_1^P - \Sigma) \left( \int_t^{t+\Delta t} K_0^P + K_1^P X_u du \right),$$

where

$$K_0^S + K_1^P X_t := \left( \rho_0^S - \rho_0^P \right) + \left( \rho_1^S - \rho_1^P \right) \cdot X_t + \Sigma \left[ (K_0^S - K_0^P) + (K_1^P - K_1^P) \cdot X_t \right] \frac{1}{\sigma_t^P \sigma_t^P} \Sigma^T.$$

Thus, changes in the log of the exchange rate are not affine in the states. However, using the following mean-preserving approximation,

$$\int_t^{t+\Delta t} K_0^P + K_1^P X_u du \approx X_{t+\Delta t} - X_t + [K_0^P + K_1^P X_t] \Delta t - K_1^P \left( e^{K_1^P \Delta t} - I \right)^{-1} (X_{t+\Delta t} - X_t) \Delta t,$$

the change in the log of the exchange rate can be written as

$$\ln S_{t+\Delta t} - \ln S_t \approx (K_0^S - \Sigma K_1^P)^\Delta t - \left\{ K_1^S K_1^P + \left( \Sigma - K_1^S K_1^P \right) K_1^P \left[ I + \left( e^{K_1^P \Delta t} - I \right)^{-1} \right] \Delta t \right\} X_t + \left\{ K_1^S K_1^P + \left( \Sigma - K_1^S K_1^P \right) K_1^P \left( e^{K_1^P \Delta t} - I \right)^{-1} \Delta t \right\} X_{t+\Delta t}.$$

Thus, changes in the log of the exchange rate are approximately affine in the states and can also be easily inverted with swap zero coupon rates to obtain implied states.

18 Since the model only describes changes in the exchange rate, and not levels, estimation requires an additional identifying restriction. For this additional restriction, I fix the initial exchange rate volatility implied by the model at 9.5%. This value was chosen as a balance between the implied volatility on short-dated options and a 6-month rolling window estimate of actual exchange rate volatility.
- 2-, 3-, 7-, and 10-year U.S. zero coupon swap rates;

When the model is estimated with exchange rate options, I also assume that the 1-, 3-, and 6-month at-the-money call options are also priced with error.

I use the following procedure to obtain historical quasi-maximum likelihood estimates:

- Randomly generate 25 sets of feasible starting parameters.
- Starting from the feasible parameters with the highest quasi-likelihood function, use a gradient search (implemented in Matlab) to obtain a maximum of the quasi-likelihood function.
- Repeat these steps 500 times to obtain a global maximum.

5 Empirical Results

Recall that equations (2), (4), and (5) in Section 2 identify a potential tension in dynamic international asset pricing models. Exchange rate volatility is given by $\sigma^S_t$, but the market prices of risk minus exchange rate volatility $\Lambda_t - \sigma^S_t$ also affect foreign asset prices (and the market prices of risk $\Lambda_t$ affect domestic asset prices). A priori, one might expect that using options to estimate the model should improve its ability to capture exchange rate volatility. However, by the same token, including options should decrease the model’s ability to capture the term structure of interest rates in both the U.S. and U.K. The relevant focus is the extent to which each of these effects occurs.

Table 1 presents the root mean squared pricing errors (expressed in basis points) for zero coupon Libor and swap rates in the U.S. and U.K. with maturities ranging from one month to ten years. The results suggest that including options in estimation has a negligible effect on the model’s ability to capture the term structure of interest rates in the U.S. and U.K. The largest pricing errors are for U.S. zero coupon swap rates with 2-years to maturity. There is a well-known hump at the 2-year horizon in U.S. interest rates that the both versions of the model do not appear to capture well. This finding is not surprising since only 3-month and 5-year interests rates are fit exactly in the model, and two-factor models that fit 3-month and 5-year interests rates also tend to have larger pricing errors at the 2-year horizon. As expected, the interest rate pricing errors are larger when the model is estimated using options, but not appreciably so. Overall, the model fits the term structure of interest in both the U.S. and U.K. rather well, whether it is estimated with or without using exchange rate options.
Table 1: Root Mean Squared Errors for Interest Rates in bps

<table>
<thead>
<tr>
<th></th>
<th>Maturity in Years</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1/12</td>
</tr>
<tr>
<td>U.S. w/o Options</td>
<td>15</td>
</tr>
<tr>
<td>U.S. w/ Options</td>
<td>17</td>
</tr>
<tr>
<td>U.K. w/o Options</td>
<td>9</td>
</tr>
<tr>
<td>U.K. w/ Options</td>
<td>10</td>
</tr>
</tbody>
</table>

This table presents the root mean squared errors in basis points for zero coupon Libor and swap rates. The first row provides cross-sectional pricing errors for U.S. dollar interest rates when the model is estimated without including options. The second row provides cross-sectional pricing errors for U.S. dollar rates when the model is estimated with exchange rate options priced with error. The third row provides cross-sectional pricing errors for U.K. pound rates when the model is estimated without including options. The fourth row provides cross-sectional pricing errors for U.K. pound rates when the model is estimated with exchange rate options priced with error.

As discussed in Section 2, to be considered as a viable candidate for satisfying the forward premium anomaly, a model must also capture exchange rate volatility. Figure 3 plots the option-implied volatility from 6-month at-the-money exchange rate options when the model is estimated with and without including options. When the model is estimated without including exchange rate options, it completely misses option prices. The mean option-implied volatility in the data over this sample period is 8.74%, whereas the corresponding mean option-implied volatility from the model is 13.35%. Moreover, option-implied volatility is itself more variable in the model than in the actual data. Stated in prices, the mean absolute relative pricing error for 6-month at-the-money options is 54.39%. By contrast, when options are included in estimation, the model better captures option prices. The mean option-implied volatility from the model is 8.71% and the mean absolute relative pricing error is only 7.89%.

Figure 4 compares the actual (not option-implied) volatility from the models with the estimate of volatility obtained from using a 6-month rolling window. When the model is not estimated with options, it also dramatically overstates exchange rate volatility. The sample annualized exchange rate volatility in the data calculated using one-week returns is

---

19 The results for 1-month and 3-month implied volatilities are virtually identical and are omitted to conserve space.

20 Other papers such as [Inci and Lu (2004)] compare the realized volatility of the fitted exchange rate returns to that of the actual exchange rate returns, but do not examine the volatility from their model of exchange rates. To emphasize this distinction, when the model in this paper is estimated without using options, the fitted values of exchange rate returns exactly match the realized exchange rate returns and therefore the volatility of the fitted exchange rate returns exactly matches that of the actual exchange rate returns. However, the exchange rate volatility in the model when it is estimated without using options is significantly higher than realized exchange rate volatility and exchange rate option prices implied by the model are also higher than actual option prices.
7.91% and the corresponding mean exchange rate volatility implied by the model is 13.87%. This result is consistent with Dewachter and Maes (2001) who also find that their model (which is estimated without using options) similarly overstates exchange rate volatility. By comparison, when options are included in estimation, the mean exchange rate volatility implied by the model is 9.65% which is much closer to (though still higher than) what is observed in the data. Thus, while the results in Table 1 suggest that including options has a negligible effect on the model’s ability to capture the term structure of interest rates in the U.S. and U.K., Figures 3 and 4 indicate that including options vastly improves the model’s ability to capture exchange rate volatility.

The forward premium anomaly relates the risk premium $\sigma^S_t \Lambda_t$ in exchange rate returns to the difference in interest rates $r_t - r^f_t$. When the model is estimated with options, the values of exchange rate volatility $\sigma^S_t$, market prices of risk $\Lambda_t$, and U.S. and U.K. short interest rates $r_t$ and $r^f_t$ successfully captures both exchange rate volatility and the term structures of interest rates (this is the first empirical pricing model to satisfy all of these criteria). Is the exchange rate risk premium in the model also consistent with the forward premium anomaly documented in Fama (1984)? Recall from equation (6) in Section 2 that Fama performs the following regression

$$\ln S_{t+\Delta t} - \ln S_t = \alpha + \beta (r_t - r^f_t) + \epsilon_t,$$

and finds that $\hat{\beta} < 0$ for most currency pairs. In the weekly data set used in this paper (August 2001 to July 2005), $\hat{\beta} = -4.50$ with a 95% confidence interval of $[-13.15, 4.15]$. Using the historical quasi-maximum likelihood estimates, I simulate four years of weekly exchange rate and interest rate data 1,000 times. The mean regression coefficient from the simulations is $\hat{\beta} = -5.11$ and the corresponding 95% confidence interval is $[-16.27, 6.41]$. Thus, the model successfully accounts for the forward premium anomaly. That is, the appreciation of the pound against the dollar over the sample period was not anomalous, but instead this appreciate was congruous with the prices of exchange rate options and the term structures of interest rates in the U.S. and U.K.

Since the exchange rate volatility implied by the model depends significantly on whether or not options are used to estimate the model, it is pertinent to examine what properties of the model change when options are included. To this end, Table 3 presents the parameters estimates when the model is estimated both with and without assuming that exchange rate options are also priced with error, and Figure 5 plots the corresponding implied latent states. Comparing the two plots in Figure 5, it appears that including options results in a parallel shift in the factors. Table 3 confirms that the parameters governing the dynamics of the
state variables do not change significantly whether the model is estimated with or without including options. Since the level of the factors changes, the parameters relating the short interest rates in the U.S. and U.K. to the factors also change accordingly. Perhaps not surprisingly, the parameters that change most significantly when options are included in estimation are those that govern exchange rate volatility. Since the dynamics of the factors do not change appreciably, and these differences in exchange rate volatility have only a small adverse effect on the model’s ability to capture the term structures of interest rates, it appears that options do in fact provide valuable information about exchange rate volatility that is simply much harder to identify using only time-series data on foreign exchange rates and interest rates in each currency.

The empirical model in the paper uses latent factors and no-arbitrage restrictions to relate exchange rate returns to the term structure of interest rates in the U.S. and U.K. Since I use interest rates and exchange rate returns to invert for the implied values of the latent states, these latent states can in turn be expressed as functions of interest rates and exchange rate returns. The yield on a $\tau$-maturity domestic zero coupon bond is

$$Y_t(\tau) = -\frac{1}{\tau}A(\tau) - \frac{1}{\tau}B(\tau) \cdot X_t,$$

and the yield on a $\tau$-maturity foreign zero coupon bond is

$$Y_t^f(\tau) = -\frac{1}{\tau}A^f(\tau) - \frac{1}{\tau}B^f(\tau) \cdot X_t.$$

Similarly, the log of the change in the exchange rate can be expressed as

$$\ln S_{t+\Delta t} - \ln S_t \approx \left( K_0^S - \Sigma K_0^P \right) \Delta t + K_1^S K_1^{p-1} (E_t \left[X_{t+\Delta t}\right] - X_t)$$

$$+ \left\{ K_1^S K_1^{p-1} + \left( \Sigma - K_1^S K_1^{p-1} \right) K_1^P \left( e^{K_1^P \Delta t} - I \right) \right\} \Delta t \left( X_{t+\Delta t} - E_t \left[X_{t+\Delta t}\right] \right),$$

where

$$K_0^S + K_1^S X_t := \left( \rho_0 - \rho_0^f \right) + \left( \rho_1 - \rho_1^f \right) \cdot X_t + \Sigma \left( [K_0^P - K_0] + (K_1^P - K_1) X_t \right) \underbrace{\sigma^2}_{\sigma^2} \sum^T.$$
Inverting this relationship yields

\[
X_{t+\Delta t} - \mathbb{E}_t [X_{t+\Delta t}] = \begin{pmatrix}
-\frac{1}{0.25} B (0.25)^T \\
-\frac{1}{0.25} B^f (0.25)^T \\
-\frac{1}{2} B (5)^T \\
B^S (\Delta t)
\end{pmatrix}^{-1} \begin{pmatrix}
Y_{t+\Delta t} (0.25) \\
Y^f_{t+\Delta t} (0.25) \\
Y_{t+\Delta t} (5) \\
\ln S_{t+\Delta t}
\end{pmatrix} - \mathbb{E}_t \begin{pmatrix}
Y_{t+\Delta t} (0.25) \\
Y^f_{t+\Delta t} (0.25) \\
Y_{t+\Delta t} (5) \\
\ln S_{t+\Delta t}
\end{pmatrix}
\]

When the model is estimated with options,

\[
\begin{pmatrix}
-\frac{1}{0.25} B (0.25)^T \\
-\frac{1}{0.25} B^f (0.25)^T \\
-\frac{1}{2} B (5)^T \\
B^S (\Delta t)
\end{pmatrix}^{-1} = \begin{pmatrix}
32.84 & -56.50 & 41.19 & -4.78 \\
-29.75 & 49.81 & -23.81 & 3.25 \\
17.05 & 16.15 & -86.12 & -15.76 \\
-23.08 & 23.12 & 5.67 & -6.81
\end{pmatrix}
\]

First, from equation (13) it is evident that all four factors affect exchange rate returns and interest rates in both the U.S. and U.K. This result contradicts the analysis in papers such as Ahn (2004) and Dewachter and Maes (2001) who impose a distinction between local factors that affect interest rates in only the U.S. or U.K., and global factors that affect interest rates in both currencies. This result also contradicts the assumption in Brandt and Santa-Clara (2002) and Dewachter and Maes (2001) that there is an important unpriced risk factor that affects exchange rate volatility but not exchange rate returns or interest rates in the U.S. or U.K.

From equation (13) also provides some interesting insights into the forward premium. For three of the four risk factors, the sign of the dependence on the 3-month interest rates in the U.S. and U.K. is opposite (i.e. positive dependence on the 3-month interest rate in the U.S. and negative dependence on the 3-month interest rate in the U.K., or vice versa). In particular, the fourth risk factor depends almost exactly on the difference between the 3-month interest rates in the U.S. and U.K. Since the risk factors themselves depend on (and are therefore correlated with) the difference in short interest rates between the two currencies, it is perhaps not surprising that the risk premium in exchange rate returns is negatively correlated with the difference in interest rates.

Finally, Figures 3 and 4 indicate that when options are assumed to be priced with error, the model captures the level of exchange rate volatility but not the dynamics. To address this shortcoming, I re-estimate the same model with the assumption that 6-month at-the-money exchange rate options are priced exactly (and exchange rate returns are now priced with error). Figure 6 plots the option-implied volatility from prices of 3-month at-the-money options (which are priced with error) and Figure 7 compares the actual (not option-implied) volatility with the estimate of volatility obtained from using a 6-month rolling window. As 22

\[\text{Since 6-month options are priced exactly by the model, I also assume that 6-month interest rates in the U.S. and U.K. are also priced exactly by the model (and 3-month interest rates are now priced with error).}\]

\[\text{For interest readers, Figure 8 plots the implied states and Table 4 provides the parameter estimates.}\]
expected, when options are priced exactly by the model it better captures the dynamics of both option-implied and actual exchange rate volatility. When 6-month options are priced exactly, the mean absolute relative pricing error for 3-month at-the-money options is only 4.01%. For comparison, the corresponding value when the model is estimated assuming that 6-month options are priced with error is 12.12% and the value when options are not included in estimate is 61.99%. Moreover, the mean exchange rate volatility implied by the model is 8.49%, which compares favourably with sample annualized exchange rate volatility in the data calculated using one-week returns which is 7.91%.

What is surprising is that when the model is estimated assuming that 6-month options are priced exactly, it also better captures the term structure of interest rates in the U.S. and U.K. Table 2 presents the root mean squared pricing errors (expressed in basis points) for zero coupon Libor and swap rates in the U.S. and U.K. with maturities ranging from one month to ten years. With the exception of one month maturities, these pricing errors are lower than their counterparts in Table 2. Thus, by better incorporating the information about volatility that is contained in option prices, the model is also able to better capture the term structures of interest rates. Moreover, when data is simulated from the model that is estimated assuming that 6-month options are priced exactly, the mean value of the regression coefficient is \( \hat{\beta} = -5.04 \) and the corresponding 95% confidence interval is \([-11.77, 0.44]\). Thus, the model still accounts for the forward premium anomaly and replicates the empirical findings in Fama (1984).

<table>
<thead>
<tr>
<th>Maturity in Years</th>
<th>1/12</th>
<th>3/12</th>
<th>6/12</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>U.S. w/ Options Exactly</td>
<td>28</td>
<td>15</td>
<td>*</td>
<td>17</td>
<td>11</td>
<td>0</td>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>U.K. w/ Options Exactly</td>
<td>16</td>
<td>8</td>
<td>*</td>
<td>14</td>
<td>13</td>
<td>13</td>
<td>14</td>
<td>13</td>
</tr>
</tbody>
</table>

Table 2: Root Mean Squared Errors for Yields in bps - Options Exactly
This table presents the root mean squared errors in basis points for zero coupon Libor and swap rates in the U.S. and U.K. when the model is estimated assuming that 6-month at-the-money exchange rate options are priced exactly.

6 Conclusion

This paper provides an arbitrage-free empirical model that generalizes previous models in the literature and successfully captures both exchange rate volatility and the term structure of interest rates in the U.S. and U.K. I show that using exchange rate options to estimate the model provides valuable information about exchange rate volatility that is much harder to identify using only time-series data on foreign exchange rates and interest rates in each currency. Using data simulated from the model, I show that it replicates the empirical
findings in [Fama, 1984] and successfully accounts for the forward premium anomaly. That is, despite the fact that U.K. interest rates were higher than U.S. interest rates over the sample period in the paper, the appreciation of the pound and against the dollar was congruous with the prices of exchange rate options and the term structure of interest rates in the U.S. and U.K.

A natural question that this paper raises is whether the empirical model is also suitable over longer time periods and for different currency pairs. The data on option-implied volatilities that is used in the paper is only available from 2001 onwards. Since the paper shows that options are vital for capturing exchange rate volatility in the model, this data restriction unfortunately prohibits a relevant empirical study with a longer sample period. However, the empirical model can be examined for additional currency pairs since data on option-implied volatilities are available for other currencies over the same sample period. This extension is the subject of current research.
References


Figure 1: U.S. and U.K. Interest Rates
This figure plots the 3-month and 5-year U.S. and U.K. zero coupon swap rates for the sample period August 15, 2001 to July 6, 2005. The swap and Libor data are from Datastream. The swap zero curve was bootstrapped under the assumption that forward rates are constant between observed swap maturities.
Figure 2: Spot Dollar/Pound Exchange Rate
This figure plots the spot dollar/pound exchange rate over the sample period August 15, 2001 to July 6, 2005. The data is from Datastream.
Figure 3: Exchange Rate Option Implied Volatility
This figure plots the implied volatility for an at-the-money option on the dollar/pound exchange rate. The price of an at-the-money option is converted to an implied volatility using Black’s formula.
Figure 4: Exchange Rate Volatility
This figure plots the volatility implied by the model estimates and 6-month rolling window volatility estimated from the dollar/pound exchange rate data.
Figure 5: Implied States
The top figure plots the implied states when the model is estimated without including exchange rate options. The bottom figure plots the implied-states when the model is estimated assuming that exchange rate options are priced with error.
\begin{tabular}{llll}
\hline
Parameter & without options & with options & Parameter & without options & with options \\
\hline
$K^p_{03}$ & 0.7991 & 0.7989 & $K_{01}$ & -0.6635 & -0.6630 \\
$K^p_{04}$ & 1.0877 & 1.0890 & $K_{02}$ & 0.0102 & -0.0621 \\
$K^p_{111}$ & -1.6790 & -1.6926 & $K_{03}$ & 1.2506 & 1.3358 \\
$K^p_{113}$ & -0.1728 & -0.1390 & $K_{04}$ & 1.1833 & 1.2394 \\
$K^p_{114}$ & -0.0654 & -0.0610 & $K_{11}$ & -0.3332 & -0.3668 \\
$K^p_{121}$ & -0.0022 & -0.0109 & $K_{113}$ & 0.0279 & -0.1011 \\
$K^p_{122}$ & -0.8544 & -0.8493 & $K_{114}$ & 0.3122 & 0.2381 \\
$K^p_{123}$ & 0.0066 & 0.0038 & $K_{121}$ & -0.0999 & -0.0899 \\
$K^p_{124}$ & -0.4206 & -0.4182 & $K_{122}$ & -0.4856 & -0.5145 \\
$K^p_{133}$ & 0.0724 & 0.0705 & $K_{123}$ & 0.0922 & 0.0957 \\
$K^p_{144}$ & -0.8858 & -0.8800 & $K_{124}$ & -0.1502 & -0.0830 \\
$\beta_{13}$ & 0.3428 & 0.3632 & $K_{133}$ & -1.2936 & -1.3699 \\
$\beta_{23}$ & 0.0014 & 0.0016 & $K_{143}$ & 0.1019 & 0.2014 \\
$\beta_{14}$ & 0.1268 & 0.1316 & $K_{144}$ & -1.0558 & -1.1057 \\
$\beta_{24}$ & 0.2059 & 0.1175 & $\Sigma_1$ & 0.0067 & 0.0019 \\
$\rho_0$ & 0.5045 & 0.6744 & $\Sigma_2$ & -0.1670 & -0.0573 \\
$\rho_{11}$ & 0.1934 & 0.2051 & $\rho^f_0$ & 0.4191 & 0.5365 \\
$\rho_{12}$ & 0.2296 & 0.2596 & $\rho^f_{11}$ & 0.1571 & 0.1664 \\
$\rho_{13}$ & 0.0709 & 0.0260 & $\rho^f_{12}$ & 0.2010 & 0.2259 \\
$\rho_{14}$ & -0.0568 & -0.0800 & $\rho^f_{13}$ & 0.0532 & 0.0172 \\
& & & $\rho^f_{14}$ & -0.0348 & -0.0487 \\
\hline
\end{tabular}

\textbf{Table 3: Parameter Estimates}

This table presents the parameter estimates for the model. For the model that was estimated without using options I assumed that the 3-month U.S. Libor rate, 5-year U.S. zero coupon swap rate, 3-month British pound Libor rate, and the change in the dollar pound exchange rate are priced correctly by the model. The remaining Libor and swap zero coupon rates were priced with error. For the model that was estimated using options I assumed that the 3-month U.S. Libor rate, 5-year U.S. zero coupon swap rate, 3-month British pound Libor rate, and the change in the dollar pound exchange rate are priced correctly by the model. I assumed that the remaining interest rates and 1-, 3-, and 6-month at-the-money exchange rate options were priced with error.
This figure plots the implied volatility for a 3-month at-the-money option on the dollar/pound exchange rate. The model price of an at-the-money is converted to an option-implied volatility using Black’s formula.
Figure 7: Exchange Rate Volatility - Options Exactly
This figure plots the 6-month rolling window volatility estimate of volatility from the dollar/pound exchange rate data, and the volatility implied by the model estimates, including the model that is estimated assuming that 6-month at-the-money options are priced exactly.
This figure plots the implied states when the model is estimated assuming that 6-month at-the-money exchange rate options are priced exactly.
<table>
<thead>
<tr>
<th>Parameter Estimate</th>
<th>Parameter Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{01}$</td>
<td>0.5251</td>
</tr>
<tr>
<td>$K_{02}$</td>
<td>0.7562</td>
</tr>
<tr>
<td>$K_{03}$</td>
<td>0.5844</td>
</tr>
<tr>
<td>$K_{04}$</td>
<td>0.6926</td>
</tr>
<tr>
<td>$K_{11}$</td>
<td>-0.3096</td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>0.1396</td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>-0.0200</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>0.4713</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>-0.2989</td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>-0.5709</td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>-0.3006</td>
</tr>
<tr>
<td>$K_{15}$</td>
<td>0.3285</td>
</tr>
<tr>
<td>$K_{16}$</td>
<td>-1.2789</td>
</tr>
<tr>
<td>$K_{17}$</td>
<td>0.0318</td>
</tr>
<tr>
<td>$K_{18}$</td>
<td>-0.0125</td>
</tr>
<tr>
<td>$K_{19}$</td>
<td>0.0021</td>
</tr>
<tr>
<td>$K_{20}$</td>
<td>0.0873</td>
</tr>
<tr>
<td>$K_{13}$</td>
<td>-0.0135</td>
</tr>
<tr>
<td>$K_{14}$</td>
<td>-1.6034</td>
</tr>
<tr>
<td>$K_{15}$</td>
<td>-2.1258</td>
</tr>
<tr>
<td>$K_{16}$</td>
<td>0.0827</td>
</tr>
<tr>
<td>$K_{17}$</td>
<td>0.5146</td>
</tr>
<tr>
<td>$K_{18}$</td>
<td>-0.4912</td>
</tr>
<tr>
<td>$K_{19}$</td>
<td>-2.3968</td>
</tr>
<tr>
<td>$K_{20}$</td>
<td>0.0024</td>
</tr>
<tr>
<td>$K_{21}$</td>
<td>-0.0086</td>
</tr>
<tr>
<td>$K_{22}$</td>
<td>0.0167</td>
</tr>
<tr>
<td>$K_{23}$</td>
<td>0.0147</td>
</tr>
<tr>
<td>$K_{24}$</td>
<td>0.0036</td>
</tr>
<tr>
<td>$K_{25}$</td>
<td>-0.0093</td>
</tr>
<tr>
<td>$K_{26}$</td>
<td>0.0109</td>
</tr>
<tr>
<td>$K_{27}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{28}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{29}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{30}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{31}$</td>
<td>0.0024</td>
</tr>
<tr>
<td>$K_{32}$</td>
<td>-0.0086</td>
</tr>
<tr>
<td>$K_{33}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{34}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{35}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{36}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{37}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{38}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{39}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{40}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{41}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{42}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{43}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{44}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{45}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{46}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{47}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{48}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{49}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{50}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{51}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{52}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{53}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{54}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{55}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{56}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{57}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{58}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{59}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{60}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{61}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{62}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{63}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{64}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{65}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{66}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{67}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{68}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{69}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{70}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{71}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{72}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{73}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{74}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{75}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{76}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{77}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{78}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{79}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{80}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{81}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{82}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{83}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{84}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{85}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{86}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{87}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{88}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{89}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{90}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{91}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{92}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{93}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{94}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{95}$</td>
<td>0.5365</td>
</tr>
<tr>
<td>$K_{96}$</td>
<td>0.0847</td>
</tr>
<tr>
<td>$K_{97}$</td>
<td>0.0928</td>
</tr>
<tr>
<td>$K_{98}$</td>
<td>0.0218</td>
</tr>
<tr>
<td>$K_{99}$</td>
<td>0.5365</td>
</tr>
</tbody>
</table>

**Table 4: Parameter Estimates - Options Exactly**

This table presents the parameter estimates when the model is estimated assuming that the 3-month U.S. Libor rate, 5-year U.S. zero coupon swap rate, 3-month U.K. Libor rate, and 6-month at-the-money exchange rate options are priced exactly. The remaining exchange rate options and Libor and swap zero coupon rates were assumed to be priced with error.
B Pricing Kernels in the Foreign Currency

This paper characterizes international asset pricing models by their joint specification for the pricing kernel $M$ (denominated in the domestic currency) and the exchange rate $S$. Previous research has instead characterized international asset pricing models by specifying the joint dynamics of the domestic and foreign pricing kernels $M$ and $M^f$. This section shows that these two approaches are equivalent.

Lemma 1. Let $S$ be the exchange rate expressed in units of domestic currency per unit of foreign currency. Let $M$ be the minimum variance nominal pricing kernel (denominated in the the domestic currency) such that the price in domestic currency of any payoff $P_T$ in domestic currency is

$$P_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} P_T \right],$$

and the price of any payoff $P^f_T$ in foreign currency exchanged to domestic currency is

$$S_t P^f_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} \left( S_T \frac{P^f_T}{S_T} \right) \right].$$

Then

$$\frac{M^f_t}{M^f_T} := \frac{M_T S_T}{M_t S_t},$$

(14)

is the minimum variance nominal pricing kernel (denominated in the foreign currency) such that

$$P^f_t = \mathbb{E}_t \left[ \frac{M^f_T}{M^f_t} P^f_T \right], \quad \text{and} \quad \frac{1}{S_t} P_t = \mathbb{E}_t \left[ \frac{M^f_T}{M^f_t} \left( \frac{1}{S_T} P_T \right) \right].$$

(15)

It is easy to verify that the pricing kernel in equation (14) satisfies the equations in (15). Let $\hat{M}$ be the minimum variance such pricing kernel. Then $M^f_t = M^f_t \xi_t$ where $\xi_t$ is a martingale that is independent of $\hat{M}_t$ and all payoffs $P^f_T$ and $P_T/S_T$. If $\xi_t$ is not constant, then $\hat{M}_t = M_t/\xi_t$ has lower variance than $M_t$, but

$$\mathbb{E}_t \left[ \frac{\hat{M}_T}{M_t} P_T \right] = S_t \mathbb{E}_t \left[ \frac{\hat{M}^f_t}{M^f_t} \frac{1}{S_T} P_T \right] = P_t,$$

and

$$\mathbb{E}_t \left[ \frac{\hat{M}_T}{M_t} S_T P^f_T \right] = S_t \mathbb{E}_t \left[ \frac{\hat{M}^f_t}{M^f_t} P^f_T \right] = S_t P^f_t.$$

Therefore $\hat{M}$ is a valid nominal domestic pricing kernel with a lower variance than $M$, which contradicts the assumption that $M$ is minimum variance nominal domestic pricing
kernel. Thus, $\xi_t$ is constant and $M^f$ is in fact the minimum variance nominal pricing kernel denominated in the foreign currency.

Lemma 1 implies that any two of $S$, $M$, and $M^f$ completely determines the third. In particular, it cannot be the case that $S_t = S_0 M^f_t / M^f_0 \xi_t / \xi_0$.

where $\xi_t$ is independent of both $M_t$ and $M^f_t$.

It is also important to note that the result in Lemma 1 only needs to hold for the minimum variance pricing kernels that price assets in both currencies. In particular, Lemma 1 does not relate the the minimum variance pricing kernels that price assets in only one currency. To illustrate this distinction, suppose that the price in domestic currency of a domestic asset follows a process of the form

$$dP_t = P_t \mu_t \, dt + P_t \sigma_t \, dW^1_t,$$

and the price in foreign currency of a foreign asset follows a similar process

$$dP^f_t = P^f_t \mu^f_t \, dt + P^f_t \sigma^f_t \, dW^2_t,$$

where $W^1$ and $W^2$ are independent Brownian motions. Then the domestic minimum variance pricing kernel that prices only domestic assets can be of the form

$$d\hat{M}_t = -\hat{M}_t \mu_t \, dt - \hat{M}_t \Lambda_t \, dW^1_t.$$

To see this more clearly, suppose that $S_t = S_0 M^f_t / M^f_0 \xi_t / \xi_0$. Then

$$E_t \left[ \frac{M_T}{M_t} \right] = E_t \left[ \frac{M_T^f}{M^f_t} \right] = E_t \left[ \frac{M_T}{M_t} \right] = E_t \left[ \frac{M_T^f}{M_t} \right] \Rightarrow E_t \left[ \frac{\xi_T}{\xi_t} \right] = 1,$$

and

$$E_t \left[ \frac{M_T^f}{M^f_t} \right] = E_t \left[ \frac{M_T}{M_t} \frac{S_T}{S_t} \right] = E_t \left[ \frac{M_T}{M_t} \right] = E_t \left[ \frac{M_T^f}{M ^f_t} \right] \Rightarrow E_t \left[ \frac{\xi_T}{\xi_t} \right] = 1.$$
and the foreign minimum variance pricing kernel that prices only foreign assets can be of the form

\[ d\hat{M}_t^f = -\hat{M}_t^f r^f dt - \hat{M}_t^f \Lambda^f_2 dW_{2t}. \]

If \( \hat{M} \) and \( \hat{M}^f \) also price assets in both currencies, then Lemma 1 implies that the dynamics of the exchange rate are

\[ dS_t = S_t \left[ r - r^f + \Lambda^2_1 \right] + S_t \Lambda_1 dW_{1t} - S_t \Lambda_2^f dW_{2t}. \]  \hspace{1cm} (16)

However, although the price in domestic currency of the domestic asset does not depend on \( W_2 \), the domestic market price of this risk need not be zero.\(^{25}\) That is, the domestic pricing kernel that prices both domestic and foreign assets can be of the form

\[ dM_t = -M_t r dt - M_t \Lambda_1 dW_{1t} - M_t \Lambda_2 dW_{2t}. \]  \hspace{1cm} (17a)

Similarly, the foreign pricing kernel that prices both domestic and foreign assets can be of the form

\[ dM_t^f = -M_t^f r^f dt - M_t^f \Lambda_1^f dW_{1t} - M_t^f \Lambda_2^f dW_{2t}. \]  \hspace{1cm} (17b)

In this more general case, Lemma 1 implies that the dynamics of the exchange rate are

\[ dS_t = S_t \left[ r - r^f + \left( \Lambda_1 - \Lambda_1^f \right) \Lambda_1 + \left( \Lambda_2 - \Lambda_2^f \right) \Lambda_2 \right] dt + S_t \left( \Lambda_1 - \Lambda_1^f \right) dW_{1t} + S_t \left( \Lambda_2 - \Lambda_2^f \right) dW_{2t}. \]  \hspace{1cm} (18)

The pricing kernels \( M \) and \( M^f \) have identical implications as \( \hat{M} \) and \( \hat{M}^f \) for prices in domestic currency of the domestic asset and prices in foreign currency of the foreign asset. However, the implied dynamics of the exchange rate in equations (18) and (16) may be vastly different. This simple example illustrates the importance of including information about the drift and/or volatility of exchange rates when estimating market prices of risk.

\(^{25}\)Intuitively, \( \hat{M} \) is the projection of \( M \) onto the space of payoffs of domestic securities. Since \( W_2 \) is orthogonal to the space of payoffs on domestic securities, \( \Lambda_2 \) is not present in the the minimum variance pricing kernel \( \hat{M} \) that prices all domestic payoffs. This does not mean that \( \Lambda_2 = 0 \), it simply means that we cannot empirically identify \( \Lambda_2 \) by only using asset prices that do not depend on \( W_2 \).
C Exchange Rate Option Pricing

This section describes the cumulant expansion technique used in this paper to efficiently compute exchange rate option prices and facilitate estimation. This technique was first introduced to option pricing by Jarrow and Rudd (1982) and was applied to swaption pricing by Collin-Dufresne and Goldstein (2002). The development in this section is a special case of the results provided in Almedia, Graveline, and Joslin (2005) for the general class of affine models, and is only included here for completeness.

Recall from Section 3 that exchange rate option prices (in domestic currency) are given by

$$E_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] = E_t \left[ \frac{M_T}{M_t} S_T \mathbb{1}_{\{S_T \geq K\}} \right] - K E_t \left[ \frac{M_T}{M_t} \mathbb{1}_{\{S_T \geq K\}} \right].$$

By the Lévy inversion formula,

$$E_t \left[ \frac{M_T}{M_t} S_T^b \mathbb{1}_{\{S_T \geq K\}} \right] = \frac{1}{2} E_t \left[ \frac{M_T}{M_t} S_T^b \right] - \frac{1}{\pi} \int_0^\infty \frac{1}{v} \text{Im} \left\{ K^iv \ E_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right] \right\} dv,$$

where

$$E_t \left[ \frac{M_T}{M_t} S_T^b \right] = e^{A^s(b-t) + B^s(b-t) \cdot X_t} S_t^b,$$

and $A^s$ and $B^s$ satisfy the Riccati ODEs (expressed in integral form)

$$B^s(\delta, \tau) = \int_0^\tau \left\{ \delta \left[ \rho_1 - \rho_1^f - \frac{1}{2} (1 - \delta) \beta^\top \Delta \left[ \Sigma \right] \Sigma^\top \right] - \rho_1 \right\} \text{du},$$

$$A^s(\delta, \tau) = \int_0^\tau \left\{ \delta \left[ \rho_0 - \rho_0^f - \frac{1}{2} (1 - \delta) \alpha^\top \Delta \left[ \Sigma \right] \Sigma^\top \right] - \rho_0 \right\} \text{du}.$$

If the model parameters are restricted so that the solutions $A^s$ and $B^s$ to the Riccati ODEs are known in closed form, then currency option valuation only requires numerical evaluation of a 1-dimensional integral. However, in the most flexible models, the Riccati ODEs must be solved numerically and thus valuing currency options using the Lévy inversion formula is not computationally efficient enough for use with model estimation.

Instead, this paper uses a more computationally efficient cumulant expansion technique to compute currency option prices. The cumulant expansion requires that we compute the Taylor series expansion of

$$E_t \left[ \frac{M_T}{M_t} S_T^{b-iv} \right] = e^{A^s(b-iv, T-t) + B^s(b-iv, T-t) \cdot X_t} S_t^{b-iv},$$

38
about \( v = 0 \). Define the cumulants \( c_m \) by

\[
c_m := \frac{\partial^m A^S (b - iv, T - t)}{\partial (iv)^m} \bigg|_{v=0} + \frac{\partial^m B^S (b - iv, T - t)}{\partial (iv)^m} \bigg|_{v=0} \cdot X_t ,
\]

so that

\[
\ln \mathbb{E}_t \left[ \frac{M_T}{M_t} \mathcal{S}_T^{b - iv} \right] = A^S (b, T - t) + B^S (b, T - t) \cdot X_t + (b - iv) \ln S_t + \sum_{m=1}^{\infty} \frac{(iv)^m}{m!} c_m .
\]

This cumulant expansion technique is especially well-suited to an affine framework because the cumulants are also affine in the state vector \( X_t \) with coefficients that again satisfy Riccati ODEs,

\[
\begin{align*}
\partial_v B^S (b, \tau) &= B^S (b, \tau) , \\
\partial_v A^S (b, \tau) &= A^S (b, \tau) , \\
\partial_v^1 B^S (b, \tau) &= \int_0^\tau \left\{ -i \left[ \rho_1 - \rho_0 \left( b - \frac{1}{2} \right) \beta^\top \Delta [\Sigma] \Sigma^\top + \beta^\top \Delta [\Sigma] B^S (b, u) \right] \right\} du , \\
\partial_v^1 A^S (b, \tau) &= \int_0^\tau \left\{ -i \left[ \rho_0 - \rho_0 \left( b - \frac{1}{2} \right) \alpha^\top \Delta [\Sigma] \Sigma^\top + \alpha^\top \Delta [\Sigma] B^S (b, u) \right] \right\} du , \\
\partial_v^2 B^S (b, \tau) &= \int_0^\tau \left\{ -i 2 \beta^\top \Delta [\Sigma] \partial_v B^S (b, u) + \left[ b \beta^\top \Delta [\Sigma] + K_0 \right] \partial_v^2 B^S (b, u) \right\} du , \\
\partial_v^2 A^S (b, \tau) &= \int_0^\tau \left\{ -i 2 \alpha^\top \Delta [\Sigma] \partial_v B^S (b, u) + \left[ b \alpha^\top \Delta [\Sigma] + K_0 \right] \partial_v^2 B^S (b, u) \right\} du ,
\end{align*}
\]

and for \( m > 2 \),

\[
\begin{align*}
\partial_v^m B^S (b, \tau) &= \int_0^\tau \left\{ -i m \beta^\top \Delta [\Sigma] \partial_v^{m-1} B^S (b, u) + \left[ b \beta^\top \Delta [\Sigma] + K_0 \right] \partial_v^m B^S (b, u) \right\} du , \\
\partial_v^m A^S (b, \tau) &= \int_0^\tau \left\{ -i m \alpha^\top \Delta [\Sigma] \partial_v^{m-1} B^S (b, u) + \left[ b \alpha^\top \Delta [\Sigma] + K_0 \right] \partial_v^m B^S (b, u) \right\} du ,
\end{align*}
\]

where \( \varphi_k^m = \binom{m}{k} \) if \( m \neq 2k \) and \( \varphi_k^m = \frac{1}{2} \binom{m}{k} \) if \( m = 2k \).

Once we have computed the cumulants, we can use the accurate approximation

\[
\mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] \approx \sum_{m=0}^{M} \left[ \chi_{m-1} \Phi_1 (-\ln K - c_1) + \chi_{m} \Phi_0 (-\ln K - c_1) \right] ,
\]

39
where

\[ \Phi_{-1}(y) = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{y^2}{2c_2}} \]

\[ \Phi_0(y) = \int_{-\infty}^{y} \Phi_{-1}(z) \, dz , \]

and the coefficients \( \chi_{-1}^{m} \) and \( \chi_{0}^{m} \) are related to the cumulants as described below. \( \Phi_{-1} \) and \( \Phi_0 \) are just the density and cumulative distribution of the Normal distribution. There exist accurate approximations to the cumulative Normal density, therefore computation of currency prices using a cumulant expansion does not require any numerical integration (aside from solving Riccati ODEs). We now turn to determining the coefficients \( \chi_{-1}^{m} \) and \( \chi_{0}^{m} \).

Define \( a_{m} \) to be the coefficients in a Taylor series expansion of

\[ e^{A^S(\mathbf{b} - iv, T-t) + B^S(\mathbf{b} - iv, T-t) \cdot X_t} S_{T}^{b - iv} e^{-c_1(iv) + \frac{1}{2} c_2(iv)^2} \]

about \( v = 0 \), so that

\[ \mathbb{E}_t \left[ \frac{M_T}{M_t} S_{T}^{b - iv} \right] = e^{c_1(iv) - \frac{1}{2} c_2 v^2} \sum_{m=0}^{\infty} a_{m} v^{m} . \]

Then

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} \mathbb{E}_t \left[ \frac{M_T}{M_t} S_{T}^{b - iv} \right] \, dv = \sum_{m=0}^{\infty} a_{m} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(z-c_1)v - \frac{1}{2} c_2 v^2} v^{m} \, dv \]

\[ = \sum_{m=0}^{\infty} a_{m} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial^{m} e^{iu - \frac{1}{2} c_2 u^2}}{\partial u^{m}} \bigg|_{u = -i(z-c_1)} \, dv \]

\[ = \sum_{m=0}^{\infty} \frac{\partial^{m}}{\partial u^{m}} \left\{ a_{m} \frac{1}{\sqrt{2\pi c_2}} e^{\frac{x^2}{2c_2}} \right\} \bigg|_{u = -i(z-c_1)} \]

\[ = \sum_{m=0}^{M} \frac{\partial^{m}}{\partial u^{m}} \left\{ a_{m} \frac{1}{\sqrt{2\pi c_2}} e^{\frac{x^2}{2c_2}} \right\} \bigg|_{u = -i(z-c_1)} \]

\[ = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(z-c_1)^2}{2c_2}} \sum_{m=0}^{M} \lambda_{m} (z - c_1)^{m} , \]

where the last line defines the coefficients \( \lambda_{m} \).
Then by the inverse Fourier transform,

$$
\mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] = \int_{-\infty}^{\ln K} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivz} \mathbb{E}_t \left[ \frac{M_T}{M_t} S_{T}^{b-iv} \right] dv \, dz
$$

$$
\approx \sum_{m=0}^{M} \lambda_m \int_{-\infty}^{-\ln K} \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{(z-c_1)^2}{2c_2}} (z - c_1)^m \, dz ,
$$


$\Phi_m (y)$ can be expressed in terms of $\Phi_{-1} (y)$ and $\Phi_0 (y)$ via the recursive relationship

$$
\Phi_{-1} (y) = \frac{1}{\sqrt{2\pi c_2}} e^{-\frac{y^2}{2c_2}}
$$

$$
\Phi_0 (y) = \int_{-\infty}^{y} \Phi_{-1} (z) \, dz ,
$$

$$
\Phi_m (y) = -c_2 \int_{-\infty}^{y} z^{m-1} d \Phi_{-1} (z)
$$

$$
= -c_2 \left[ y^{m-1} \Phi_{-1} (y) - (m - 1) \Phi_{m-2} (y) \right] .
$$

Therefore,

$$
\mathbb{E}_t \left[ \frac{M_T}{M_t} (S_T - K)^+ \right] \approx \sum_{m=0}^{M} \left[ \chi_{-1}^m \Phi_{-1} (-\ln K - c_1) + \chi_0^m \Phi_0 (-\ln K - c_1) \right] ,
$$

as desired.

Finally, $M$ must be chosen to balance accuracy and computational speed. I follow Collin-Dufresne and Goldstein (2002) and choose $M = 7$. 

41